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THESIS

A FURTHER COMPARISON OF DETERMINISTIC AND
STOCHASTIC LANCHESTER-TYPE COMBAT MODELS

by

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This thesis examines the differences of deterministic and stochastic LANCHESTER-type combat models. Using an example of square-law attrition, solution methods and solutions are described. A new analytic solution for equal attrition rate coefficients is given. The numerical comparison includes hypotheses about the expected force levels and the variability in the expected force levels as a function of time, initial force levels, and breakpoint force levels.

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A Further Comparison of Deterministic and Stochastic Lanchester-Type Combat Models

by

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ABSTRACT

This thesis examines the differences of deterministic and stochastic LANCHESTER-type combat models. Using an example of square-law attrition, solution methods and solutions are described. A new analytic solution for equal attrition rate coefficients is given. The numerical comparison includes hypotheses about the expected force levels and the variability in the expected force levels as a function of time, initial force levels, and breakpoint force levels.

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I. INTRODUCTION

Combat models are widely used as decision aids in the defense-planning process, at least within the NATO alliance. Current operational combat models are very complex because combat is a very complex process. Unfortunately it is difficult (if not impossible) for the beginner to understand the modelling approaches, concepts and motivation, that may have been used to build such operational models. However, one frequently considers a simple model as a paradigm for the development and understanding of such complex models. This basic approach will be used in this thesis to explore certain issues in the on-going debate about the relative merits of stochastic and deterministic combat models.

A simple model is examined to explore differences between a deterministic and a stochastic approach to a certain type of analytical combat model. As already mentioned, combat is a very complex process, but it is also a complex <u>random</u> process, which can be supported by many examples from military history. Analytical models are abstractions and very often simplifications of reality. It seems to be a legitimate question to ask, what effects the further abstraction of neglecting the randomness in combat may have. At this moment, it should be pointed out that within existing operational analytical models, both stochastic and deterministic models are used.

Previous work done by SPRINGALL [9] and CLARK [4] evolved around theoretical aspects. Their main concern was to give exact analytical solutions and their proofs. CRAIG [5] started to explore the differences

between stochastic and deterministic models more from the numerical point of view, which will be continued in this thesis.

In the next chapter, a deterministic and stochastic version of a differential combat model will be described. The deterministic versions are well-known as LANCHESTER's equations of modern warfare, which were developed in 1914. Combat models, which model attrition from enemy action through a system of differential equations, are usually referred to as LANCHESTER-type models of warfare.

II. THE PARADIGM MODELS

A. THE DETERMINISTIC MODEL

First, LANCHESTER's equations of modern warfare will be briefly reviewed and some simple extensions given.

In 1914 LANCHESTER [7] hypothesized that under "modern conditions" in a combat between two homogeneous forces the firepower of the surviving weapons of one side can be concentrated on the surviving targets of the other side, so that each side's casualty rate is proportional to the number of enemy firers. This can be described by the following equations:

$$\frac{dx}{dt} = -ay \tag{2.1}$$

$$\frac{dy}{dt} = -bx \tag{2.2}$$

with initial conditions

$$X(0) = x_0 \tag{2.3}$$

$$Y(0) = y_0 \tag{2.4}$$

where a is the attrition rate with which the Y-force attrits the X-force, similarly for b. X_0 and Y_0 are the initial force levels and X(t) and Y(t) are the force levels at time t. The force levels, as a function of time t, can be written as

$$X(t)=x_0\cosh(\sqrt{ab}\ t)-\sqrt{a/b}\ y_0\sinh(\sqrt{ab}\ t) \tag{2.5}$$

$$Y(t)=y_0\cosh(\sqrt{ab}\ t)-\sqrt{b/a}\ x_0\sinh(\sqrt{ab}\ t) \tag{2.6}$$

The state equation relating initial force levels with force levels at some time t can be derived by dividing (2.1) by (2.2), separating

variables and integrating to yield

$$b(x_0^2-X(t)^2)=a(y_0^2-Y(t)^2). \tag{2.7}$$

This form of the state equation explains why this model is also referred to as the "square-law" attrition process. WEISS [11] has given a set of assumptions under which LANCHESTER's equations for modern warfare may apply:

A1.) Two homogeneous forces are engaged in combat. Every unit on a particular side has the same capabilities. The attrition rate may be different for the two forces.

A2.) Each unit on one side is within weapon range of all units on the other side.

A3.) The effects of successive rounds on the target are independent.

A4.) Each unit has perfect knowledge of target locations and fires only at live target (one at a time) killing them at a constant rate, which does not depend on the number of targets alive.

A5.) Fire is uniformly distributed over surviving targets.

The above model implies a fight until one force is annihilated. Therefore the model will be slightly changed by introducing the concept of unit breakpoints, X_{bp} and Y_{bp}, which are simply force levels at which the side, who reaches it first "breaks off" the engagement, leaving the other side as a winner. Also, to be more precise, it should be noted that negative force levels for breakpoints equal zero or force levels

less than nonzero breakpoints are impossible. So the deterministic LANCHESTER-type combat model with "square-law" attrition takes the following form:

$$\frac{dx}{dt} = \begin{cases} -ay & x_{bp} < X(t) \\ & y_{bp} < Y(t) \\ 0 & \text{otherwise} \end{cases} \tag{2.8}$$

$$\frac{dy}{dt} = \begin{cases} -bx & x_{bp} < X(t) \\ & y_{bp} < Y(t) \\ 0 & \text{otherwise} \end{cases} \tag{2.9}$$

with initial conditions

$$X(0) = x_0 \tag{2.10}$$

$$Y(0) = y_0 \tag{2.11}$$

The model in this form, equations (2.8) through (2.11), was used for comparisons throughout the thesis.

B. THE STOCHASTIC MODEL

There are several ways to include random variations in LANCHESTER-type models. These are:

* The attrition rate coefficients may be random variables.

* The enemy's initial force level may be a random variable, weakening the assumption of perfect knowledge.

* The breakpoints may be random variables.

* The casualty rate is fixed, but the occurrence of casualties over time may be random.

The only random variation considered here will be the random occurrence of casualties over time. Another specification was to choose a

model similar to the "square-law"attrition in order to allow comparisons. In other words, the question to be asked is "how do random fluctuations in the occurrence of casualties modify the deterministic results of the square-law attrition process?"

The approach used here was a continuous parameter MARKOV chain model, where the time t varies continuously and the number of combatants on each side is a non-negative integer. Let M(t) be the size of the X-force at time t with a particular state value m. Let N(t) be the size of the Y-force at time t with a particular state value n. Let m_0 and n_0 be the initial force levels and m_{bp}, n_{bp} be the breakpoint force levels of X and Y respectively. Fig. 1 shows the state space of this MARKOV chain model. Note that at a given time t, any state is described by the two force levels of the X and Y force. As each side loses units due to attrition and no replacements are allowed, it is easy to understand why BILLARD [1] referred to this type of process as a "bivariate death process."

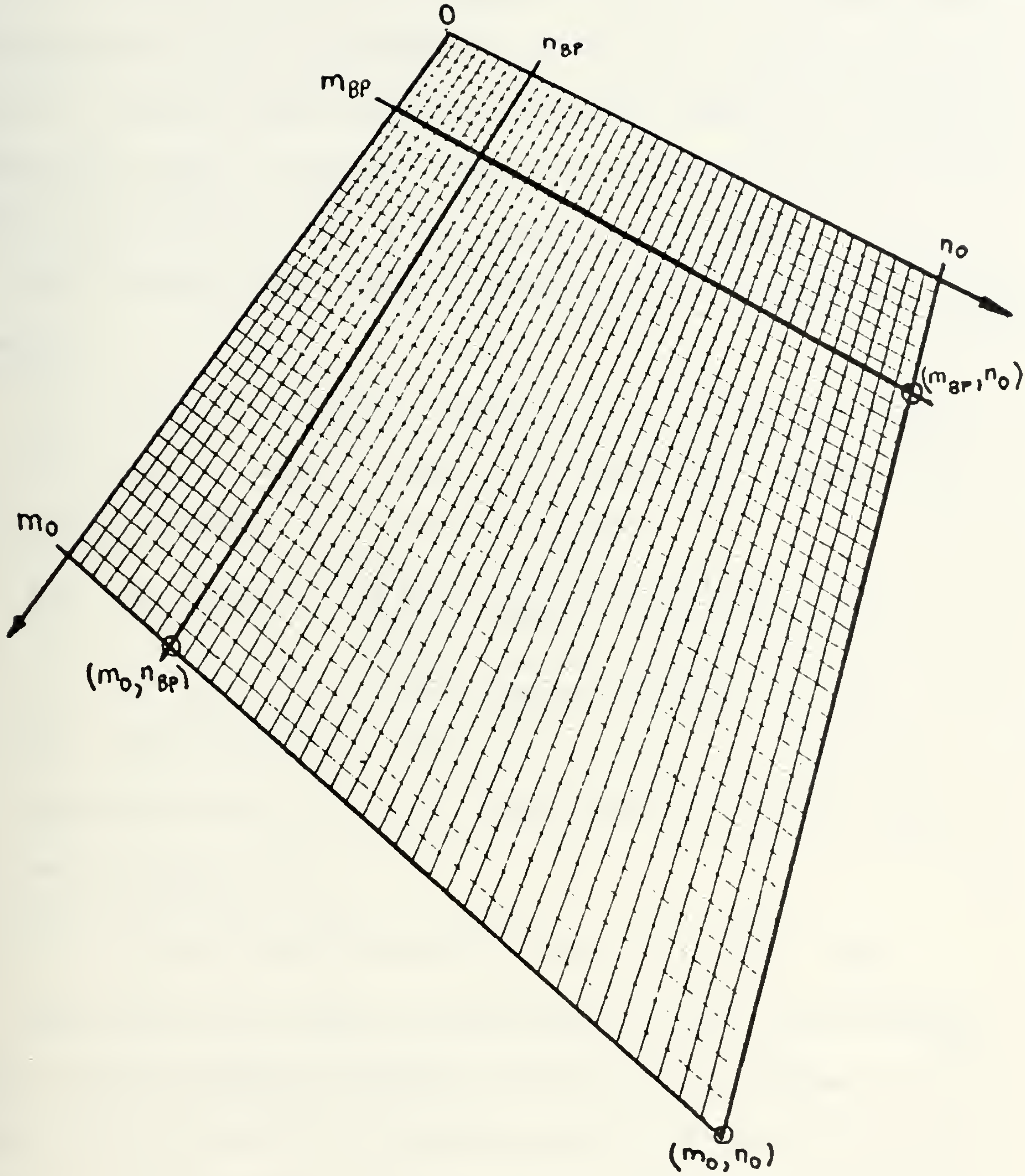

Figure 1 - STATE SPACE OF THE MARKOV CHAIN MODEL

For the description of the stochastic square-law attrition process corresponding to the two deterministic differential equations (2.8) and (2.9), a system of many differential equations, depending on the battle termination model, is required. This system will be given for a fixed-force-level-breakpoint battle with square-law attrition.

The following assumptions yield the stochastic square-law attrition process.

A1.) The attrition process depends only on the current system state and time and not on the past history (this assumption is usually referred to as markovian property).

A2.) The probability $P\left(\begin{matrix}\text{one x casualty during the}\\ \text{time interval t to t+h}\end{matrix}\right) = ah$

A3.) The probability $P\left(\begin{matrix}\text{one y casualty during the}\\ \text{time interval t to t+h}\end{matrix}\right) = bh$

A4.) The probability of more than one casualty occurring in the time interval t to t+h is of the order of magnitude $o(h)$, where $\lim_{h\to 0} o(h)/h = 0$.

A5.) No more casualties can occur once $m = m_{bp}$ or $n = n_{bp}$.

Making the time interval h infinitesimally small, the following set of forward CHAPMAN-KOLMOGOROV equations can be developed. Let $P(t,m,n)$ be the probability that the system is in state (m,n) at a time t. For convenience each equation is related to a region in the state space shown in Fig. 2.

For $m=m_0$ and $n=n_0$, Region I

$$\frac{dP}{dt}(t,m_0,n_0) = -(an_0+bm_0)\ P(t,m_0,n_0) \tag{2.12}$$

for $m_{bp}<m<m_0$ and $n=n_0$, Region II

$$\frac{dP}{dt}(t,m,n_0) = an_0\ P(t,m+1,n_0)-(an_0+bm)\ P(t,m,n_0) \tag{2.13}$$

for $n_{bp}<n<n_0$ and $m=m_0$, Region III

$$\frac{dP}{dt}(t,m_0,n) = bm_0\ P(t,m_0,n+1)-(an+bm_0)\ P(t,m_0 n) \tag{2.14}$$

for $m_{bp}<m<m_0$ and $n_{bp}<n<n_0$, Region IV

$$\frac{dP}{dt}(t,m,n) = anP(t,m+1,n)+bmP(t,m,n+1)-(an+bm)P(t,m,n) \tag{2.15}$$

for $m=m_{bp}$ and $n_{bp}<n<n_0$, Region V

$$\frac{dP}{dt}(t,m_{bp},n) = anP(t,m_{bp}+1,n) \tag{2.16}$$

for $n=n_{bp}$ and $m_{bp}<m<m_0$, Region VI

$$\frac{dP}{dt}(t,m,n_{bp}) = bmP(t,m,n_{bp}+1) \tag{2.17}$$

for $m=m_{bp}$ and $n=n_{bp}$, Region VII

$$P(t,m_{bp},n_{bp}) = 0 \text{ for all } t \tag{2.18}$$

because of the definition of a breakpoint force level. The initial condition is

$$P(0,m_0,n_0) = 1.0 \tag{2.19}$$

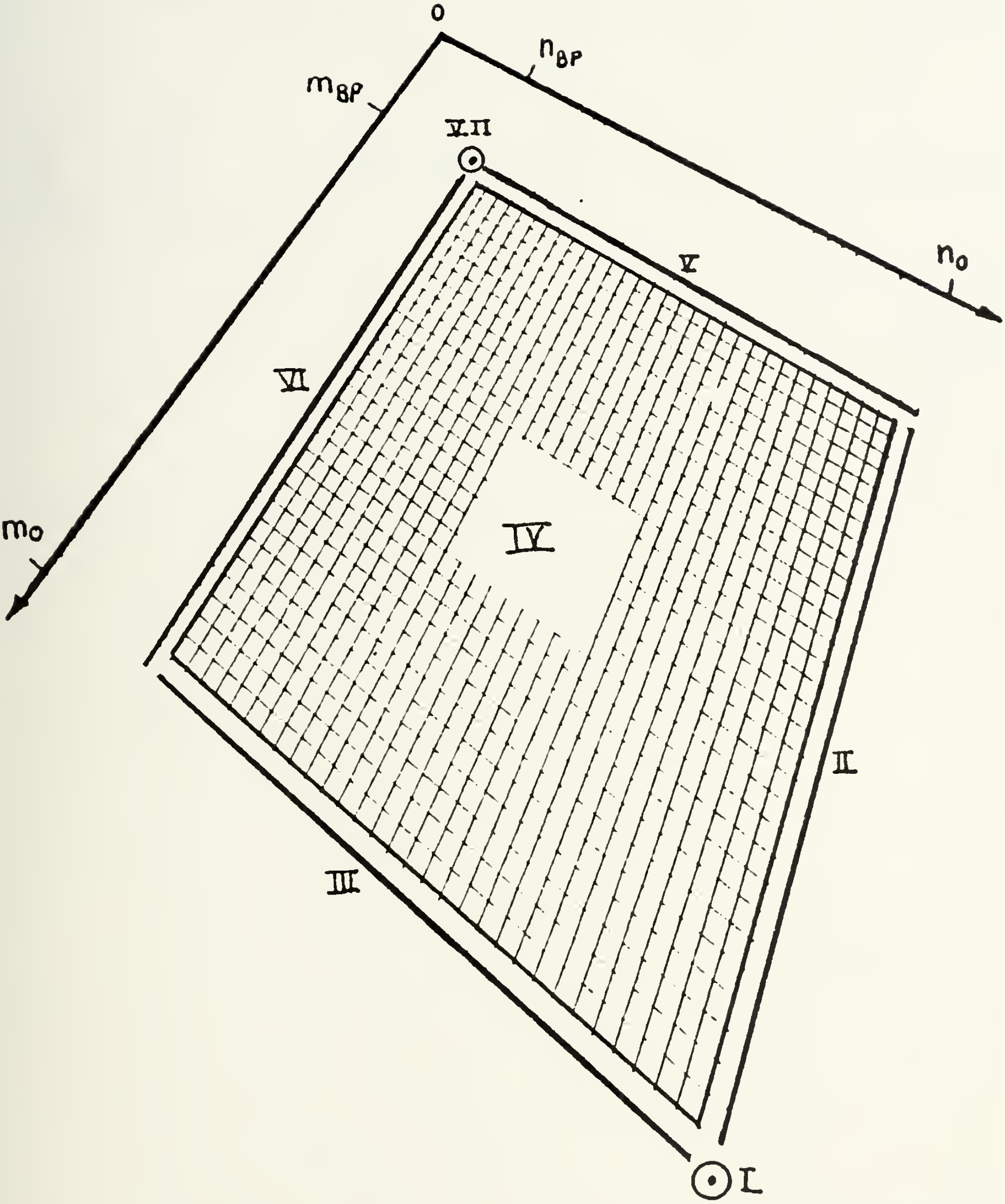

Figure 2 - REGIONS IN THE STATE SPACE

As P(t,m,n) is a joint probability distribution, the following must also be true

$$0 \leq P(t,m,n) \leq 1.0 \tag{2.20}$$

and

$$\sum_{m=m_{bp}}^{m_0} \sum_{n=n_{bp}}^{n_0} P(t,m,n) = 1.0 \tag{2.21}$$

III. SOLVING THE DIFFERENTIAL EQUATIONS

A. SOLUTIONS FOR THE DETERMINISTIC MODEL

Force levels as a function of time were already given in equations (2.5) and (2.6). It is relatively easy to obtain analytical solutions for such simple deterministic models as described before. On the other hand, it is necessary to point out that for models with any degree of operational realism, analytical methods for solving the differential equations are usually not available. Therefore, some numerical method with a digital computer is usually used. TAYLOR [10] has summarized in his Appendix C the most widely used numerical methods, a discussion of which seems unnecessary at this point.

B. SOLUTIONS TO THE STOCHASTIC MODEL

Even for this relatively simple stochastic model with fixed force level breakpoints, which are usually nonzero, a complete set of general analytical solutions for the differential equations (2.12) through (2.19) has not been found. On the other side, given some minor restrictions like a fight to the finish or equal attrition rate coefficients, solutions, or at least solution methods have been proposed which will be briefly discussed in the next section.

First, the method for getting the state probabilities used here will be described. Numerical solutions were obtained using the fourth-order RUNGE-KUTTA method, which is probably one of the best known finite difference approximations to ordinary differential equations (next to the

EULER-CAUCHY-method). To increase the accuracy of the overall solutions analytical results for certain regions of the state space were substituted. These analytical solutions will be stated now. For region I, i.e. no casualties on either side, the solution to (2.12) is

$$P(t,m_0,n_0) = \exp \ -(an_0+bm_0)t \tag{3.1}$$

which can be derived by the standard method used for this kind of differential equation. For the boundary cases, region II and III, i.e. one of the two sides has not yet had a casualty, TAYLOR [10] has given the analytical expression as

<u>for $m_{bp}<m<m_0$ and $n=n_0$, Region II</u>

$$P(t,m,n_0) = \frac{1}{J!}\left\{an_0/b(e^{bt}-1)\right\}^J \exp(-(bm_0+an_0)\ t)\ , \tag{3.2}$$

where $J = m_0-m$

<u>for $n_{bp}<n<n_0$ and $m=m_0$, Region III</u>

$$P(t,m_0,n) = \frac{1}{K!}\left\{bm_0/a(e^{at}-1)\right\}^K \exp(-(bm_0+an_0)\ t)\ , \tag{3.3}$$

where $K = n_0-n$.

These two equations (3.2) and (3.3) were obtained by recursively solving equations (2.12), (2.13) and (2.14) "from the top down."

C. OTHER ANALYTICAL SOLUTIONS

The solutions or solution methods for getting the state probabilities will only be stated for the square-law attrition process. Only two were used for the numerical work for this thesis.

Apparently, one of the "oldest" analytic solutions was given by BROWN [3] in 1955 for the general stochastic LANCHESTER-type combat model with time independent attrition rates. His approach and solution will be briefly outlined for square-law attrition. Consider a path from state (m_0 , n_0) to some state (m,n). This path can be described as a sequence of $J = (m_0 - m)$ zeros and $K = (n_0 - n)$ ones, where a zero corresponds to an X casualty and a one to a Y casualty. Using the binary representation of a positive integer, one can define to each realization of a battle path an integer k given by

$$k = d_{k,1} d_{k,2} \cdots d_{k,J+K}$$

where $d_{k,r} = 1$ if the r^{th} casualty along a battle path corresponding to k is a Y casualty and $d_{k,r} = 0$ otherwise. Also let $I_{J,K}$ be the set of all positive integers whose binary representation contains exactly K ones and J zeros. Then

$$m_{k,r} = m_0 - r + \sum_{j=1}^{r} d_{k,j}$$

$$n_{k,r} = n_0 - \sum_{j=1}^{r} d_{k,j} .$$

Then BROWN [3] has shown that

$$P(t,m,n) = \frac{1}{2\pi} \sum_{k \varepsilon I_{J,K}} \int_{-\infty}^{\infty} \prod_{r=0}^{J+K-1} K_{k,r}$$

$$\cdot \quad \frac{\exp(-iut)-\exp(-tl(m_{k,r},n_{k,r}))}{l(m_{k,r},n_{k,r})-iu} \quad du \quad , \quad (3.4)$$

where $i = \sqrt{-1}$,

$$l(m,n) = an+bm,$$

$$K_{k,r} = \frac{g_{k,r+1}}{1-iu/l(m_{k,r},n_{k,r})}$$

and

$$g_{k,r+1} = d_{k,r+1}an_{k,r}+(1-d_{k,r+1})bm_{k,r} \; .$$

There was no indication that nonzero breakpoints were excluded. A discussion of this solution follows in the next section.

About 14 years later, in 1969, CLARK [4] proposed another approach which TAYLOR [10] called a "hybrid analytical-numerical" method. The restriction is that the breakpoints have to be zero, i.e. it is a fight to the finish. Although proposed for a general time independent attrition function, this approach will be outlined for "square-law" attrition. Then according to CLARK [4] the state probabilities are given by

<u>for $0<m\leq m_0$ and $0<n\leq n_0$</u>

$$P(t,m,n) = \sum_{j=m}^{m_0} \sum_{k=n}^{n_0} C_{j,k}^{m,n} \cdot \exp\left(-(ak+bj)t\right) \quad , (3.5)$$

<u>for $0<m\leq m_0$ and $n=0$</u>

$$P(t,m,0) = C_{0,0}^{m,0} + \sum_{j=m}^{m_0} \sum_{k=1}^{n_0} C_{j,k}^{m,0} \cdot \exp\left(-(ak+bj)t\right), \quad (3.6)$$

<u>for $m=0$ and $0<n\leq n_0$</u>

$$P(t,0,n) = C_{0,0}^{0,n} + \sum_{j=m}^{m_0} \sum_{k=1}^{n_0} C_{j,k}^{0,n} \cdot \exp\left(-(ak+bj)t\right), (3.7)$$

and at last there is to remember that $P(t,0,0) = 0$ for all times t. The constants $C_{j,k}^{m,n}$ are determined by a system of partial difference equations.

<u>For $0<m<j\leq m_0$ and $0<n<k\leq n_0$</u>

$$C_{j,k}^{m,n} = \frac{an\ C_{j,k}^{m+1,n} + bm\ C_{j,k}^{m,n+1}}{a(n-k)+b(m-j)} \quad , (3.8)$$

<u>for $0<m<j\le m_0$ and $0<n=k\le n_0$</u>

$$C^{m,n}_{j,n} = \frac{an\, C^{m+1,n}_{j,n}}{b(m-j)} \qquad , \qquad (3.9)$$

<u>for $0<m=j\le m_0$ and $0<n<k\le n_0$</u>

$$C^{m,n}_{m,k} = \frac{bm\, C^{m,n+1}_{m,k}}{a(n-k)} \qquad , \qquad (3.10)$$

<u>for $0<m=j\le m_0$ and $0<n=k\le n_0$</u>

<u>but $(m,n) \neq (m_0,n_0)$</u>

$$C^{m,n}_{m,n} = -\sum_{j=m}^{m_0} \sum_{k=n+1}^{m_0} C^{m,n}_{j,k} - \sum_{j=m+1}^{m_0} C^{m,n}_{j,k} \qquad , \qquad (3.11)$$

with $C^{m_0,n_0}_{m_0,n_0} = 1.0.$

Also for $0<m\leq j\leq m_0$ and $0=n<k\leq n_0$

$$C_{j,k}^{m,0} = -\frac{bm\, C_{j,k}^{m,1}}{ak+bj} \quad , \qquad (3.12)$$

similarly for $0=m<j\leq m_0$ and $0<n\leq k\leq n_0$

$$C_{j,k}^{0,n} = -\frac{an\, C_{j,k}^{1,n}}{ak+bj} \quad . \qquad (3.13)$$

Then for $1\leq m\leq m_0$

$$C_{0,0}^{m,0} = -\sum_{j=m}^{m_0} \sum_{k=1}^{n_0} C_{j,k}^{m,0} \quad , \qquad (3.14)$$

and finally for $1\leq n\leq n_0$

$$C_{0,0}^{0,n} = -\sum_{j=1}^{m_0} \sum_{k=n}^{n_0} C_{j,k}^{0,n} \quad . \qquad (3.15)$$

Though having the publishing date of 1979, the next approach was published in June 1980 by BILLARD [1]. She considered the LANCHESTER-type square-law attrition combat model as a pure death-process and applied SEVERO's [8] recursive theorem for solving differential equations. As before, only a fight to the finish has been considered.

The first step is to identify each point (m,n) in the state space by a counting coordinate k, where

$$k = (m_0+1)(n_0+1)$$
$$-m(n_0+1)-n \quad . \qquad (3.16)$$

Then

$$P[t,m,n]= P[t,k] \qquad (3.17)$$

and the differential equations (2.12) through (2.18) take on a slightly different form. As an example, (2.15) will be given by

$$\frac{dP}{dt}(t,k) = anP[t,k-n_0-1]+bmP[t,k-1]$$
$$-(an+bm)P[t,k] \quad . \qquad (3.18)$$

The whole set of differential equations was then expressed in matrix terms as

$$\frac{d}{dt}\underline{P}(t) = \underline{BP}(t) \qquad (3.19)$$

with a solution given as

$$\underline{P}(t) = \underline{Ce}\ (t) \qquad (3.20)$$

where $\underline{e}(t)$ is the $(m_0+1)(n_0+1)\text{x}1$ -vector with elements $\exp(b_k t)$ with b_k being the k^{th} diagonal element of the matrix $\underline{B}$. The matrix $\underline{B}$ can be partitioned into submatrices, whose m-coordinate is common, due to the ordering defined by the counting coordinate k (Equation 3.16).

Then

$$\underline{B} = (\underline{b}_{uv}),\ u,v = 1,2...m_0+1$$

where the submatrices $\underline{b}_{uv}$ have the elements

$$\underline{b}_{uv} = (b_{uv}(p,q)), \quad p,q = 1,2...n_0+1.$$

So the matrix $\underline{B}$ has the elements

$$b_{uu}(p,p) = -a(n_0-p+1)-b(m_0-u+1)$$
$$\text{for } u = 1,2...m_0 \text{ and } p = 1,2...n_0$$

$$b_{uu}(p,p-1) = b(m_0-u+1) \text{ for } u = 1,2...m_0+1$$
$$p = 2,3...n_0+1$$

and

$$b_{u,u-1}(p,p) = a(n_0-p+1) \text{ for } u = 2,3...m_0+1$$
$$P = 1,2...n_0+1.$$

All other elements are zero.

Thus, the matrix $\underline{B}$ has at most 3 nonzero entries per row or column. The matrix $\underline{C}$ can be partitioned in the same way. Then using SEVERO's [8] theorem and the special form of the matrix $\underline{B}$, only a part of the $\underline{C}$-matrix needs to be determined. This part will be omitted here, but the final result will be given by

$$P(t,k) = \sum_{j=1}^{k} c(k,j)\cdot\exp(b_j t) \tag{3.21}$$

where $c(k,j)$ is the $(k,j)^{th}$ element of the solution matrix $\underline{C}$.

The previous two approaches have required that the force level breakpoints be zero. Now, a result will be given whose restriction is that the attrition rate coefficients be equal, but nonzero breakpoints are allowed. For further reference it will be called the Equal-Attrition-Rate-Coefficient-Solution (EARCS).

Let $a = b = f$.

For $m_{bp}<m\leq m_0$ and $n_{bp}<n\leq n_0$

$$P(t,m,n) = \frac{C(m,n)}{(m_0+n_0-m-n)!}$$

$$\cdot(1-e^{-ft})^{m_0+n_0-m-n} \cdot\exp(-f(m+n)t) \tag{3.22}$$

for $m = m_{bp}$ and $n_{bp}<n\leq n_0$

$$P(t,m_{bp},n) = \frac{fn \quad C(m_{bp}+1,n)}{J!} \cdot$$

$$\cdot\sum_{k=0}^{J} (-1)^k \binom{J}{k} \left\{\frac{1-\exp\left(-ft(m_{bp}+1+n+k)\right)}{f(m_{bp}+1+n+k)}\right\}, \tag{2.23}$$

where $J = m_0+n_0-m_{bp}-1-n$

for $n = n_{bp}$ and $m_{bp}<m\leq m_0$

$$P(t,m,n_{bp}) = \frac{fm \quad C(m,n_{bp}+1)}{K!}$$

$$\cdot\sum_{j=0}^{K} (-1)^j \binom{K}{j} \left\{\frac{1-\exp\left(-ft(m+1+n_{bp}+j)\right)}{f(m+1+n_{bp}+j)}\right\}, \tag{3.24}$$

where $K = m_0+n_0-m-n_{bp}-1$.

The coefficients $C(m,n)$ satisfy for $m_{bp}<m<m_0$ and $n_{bp}<n<n_0$ the following partial difference equation

$$C(m,n) = nC(m+1,n) + mC(m,n+1) \tag{3.25}$$

with the boundary conditions

$$C(m,n_0) = (n_0)^{m_0-m} \quad \text{for } m_{bp}<m\leq m_0$$

and

$$C(m_o,n) = (m_0)^{n_0-n} \quad \text{for } n_{bp}<n\leq n_0$$

This result has been developed using a method verbally proposed by TAYLOR. The method will now be outlined. ISBELL and MARLOW [6] described a stochastic LANCHESTER-type attrition process with a different attrition function. Instead of the attrition of one force being proportional to the number of of enemies of the opposing force as in the square-law attrition (e.g. for the attrition of the M-force let

$$A(m,n) = an$$

be the attrition rate and

$$B(m,n) = bm$$

the attrition rate for the N-force with square-law attrition), their attrition rates looked like

$$A(m,n) = an+cm$$

and

$$B(m,n) = bm+dn$$

with the restriction that

$$a+c = b+d.$$

But with $c=d=0$ and $a=b=f$ we are back to square-law attrition. This leads to equations (3.22) and (3.25). Equations (3.23) and (3.24) were derived in the following way (e.g. for 3.23). Solving equation (3.22) for $m = m_{bp}+1$ yields

$$P(t,m_{bp},n) = \frac{C(m_{bp}+1,n)}{(m_0+n_0-m_{bp}-1-n)!} \cdot \exp\left(-ft(m_{bp}+1+n)\right)$$

$$\cdot (1-e^{-ft})^{m_0+n_0-m_{bp}-1-n} \tag{3.26}$$

substituting for the second factor its BINOMIAL expansion

$$(1-e^{-ft})^J = \sum_{k=0}^{J} (-1)^k \binom{J}{k} e^{-ftk}(1)^{J-k}$$

with $J = m_0+n_0-m_{bp}-1$ and multiplying through by the third factor. Then using equation (2.16), the differential equation for $m = m_{bp}$ and $n_{bp}<n\leq n_0$, and substituting equation (3.26) into the extended form, it can now easily be integrated to yield equation (3.23).

D. DISCUSSION

In the discussion of the analytical solutions outlined in the last section, there is one important point. BROWN [3] himself points out that unless m is close to m_0 and n is close to n_0 his result (equation (3.4)) is of "little practical interest." Most of the analytical solutions, especially for more general LANCHESTER-type models, have little more than "symbolic" character. BROWN's solution is a good example of that.

In comparing the solutions given by BILLARD [2] and CLARK [4], this author has the feeling that both solutions are equivalent and only the representation is different. This

intuitive guess needs verification. It may be concidence that CLARK [4] and SEVERO [8] published their work in the same year.

The last presented solution (equation (3.22) through (3.24)) seems to be relatively handy for use on a computer. It has a big advantage over numerical solution methods other than its accuracy, because it is an exact result. Like CLARK's method, the coefficients have to be calculated only once for a given set of input data. Then, to get the state probabilities for a certain point in time you have to make only one set of calculations, as opposed to the numerical methods where one has to go from time $t = 0$ to time $t = t$ in small time steps and then have only an approximate result.

E. IMPORTANCE OF THE STATE PROBABILITIES

The state probabilities as a function of time are the key to calculating several quantities of interest. These are expected force levels as a function of time, variances and standard deviations in the force levels and also the probability of winning. These quantities are necessary to legitimately compare the stochastic with the deterministic results.

To get at least a feeling of how the state probabilities evolve over time, the joint probability distribution will be presented in a 3-D-picture. It is indeed surprising that more use has not been made of computer graphics to

investigate the dynamics of a stochastic LANCHESTER-type combat model. Table 1 gives the data used for the next five figures. These figures may be thought of as "snapshots" of the joint probability for the survivors in this battle taken at a sequence of increasing times.

TABLE 1

Data for the Numerical Example 1

Force Levels	$m_0 = 40$	$m_{bp} = 0$
	$n_0 = 40$	$n_{bp} = 0$
Attrition Rates	$a = 0.008$	M casualties per minute and N firer
	$b = 0.004$	N casualties per minute and M firer
At Times	$t_1 = 0.025\ t_f$	
	$t_2 = 0.250\ t_f$	
	$t_3 = 0.500\ t_f$	
	$t_4 = 0.750\ t_f$	
	$t_5 = 1.000\ t_f$	

where $t_f = 155.81$ minutes is the time a deterministic battle with the same force levels and unit breakpoints ends, i.e. $x(t_f) = 0.0$ and $y(t_f) = 28.28$

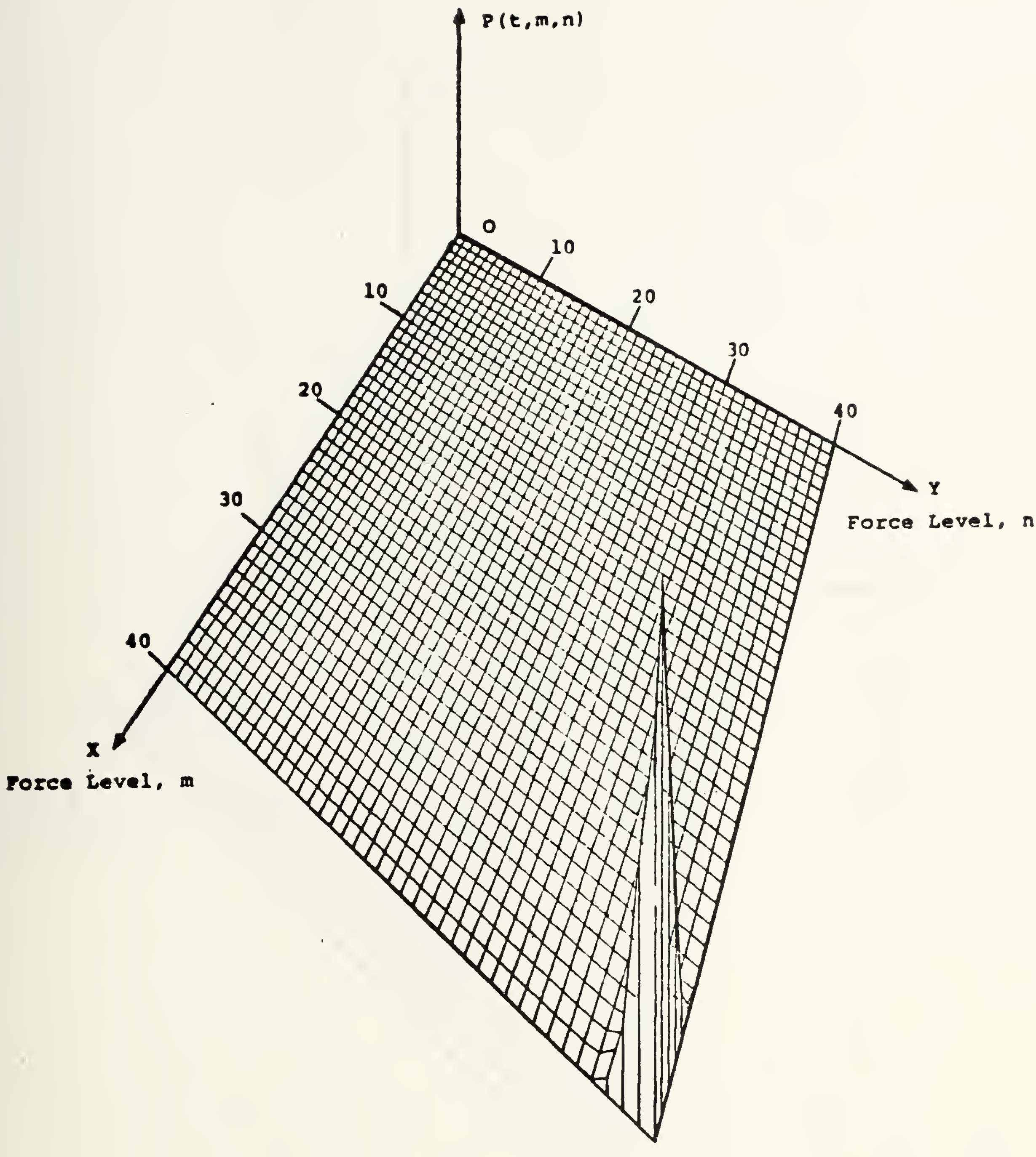

Figure 3 - PLOT OF JOINT PROBABILITIES P(t,m,n)

with data according to Table 1

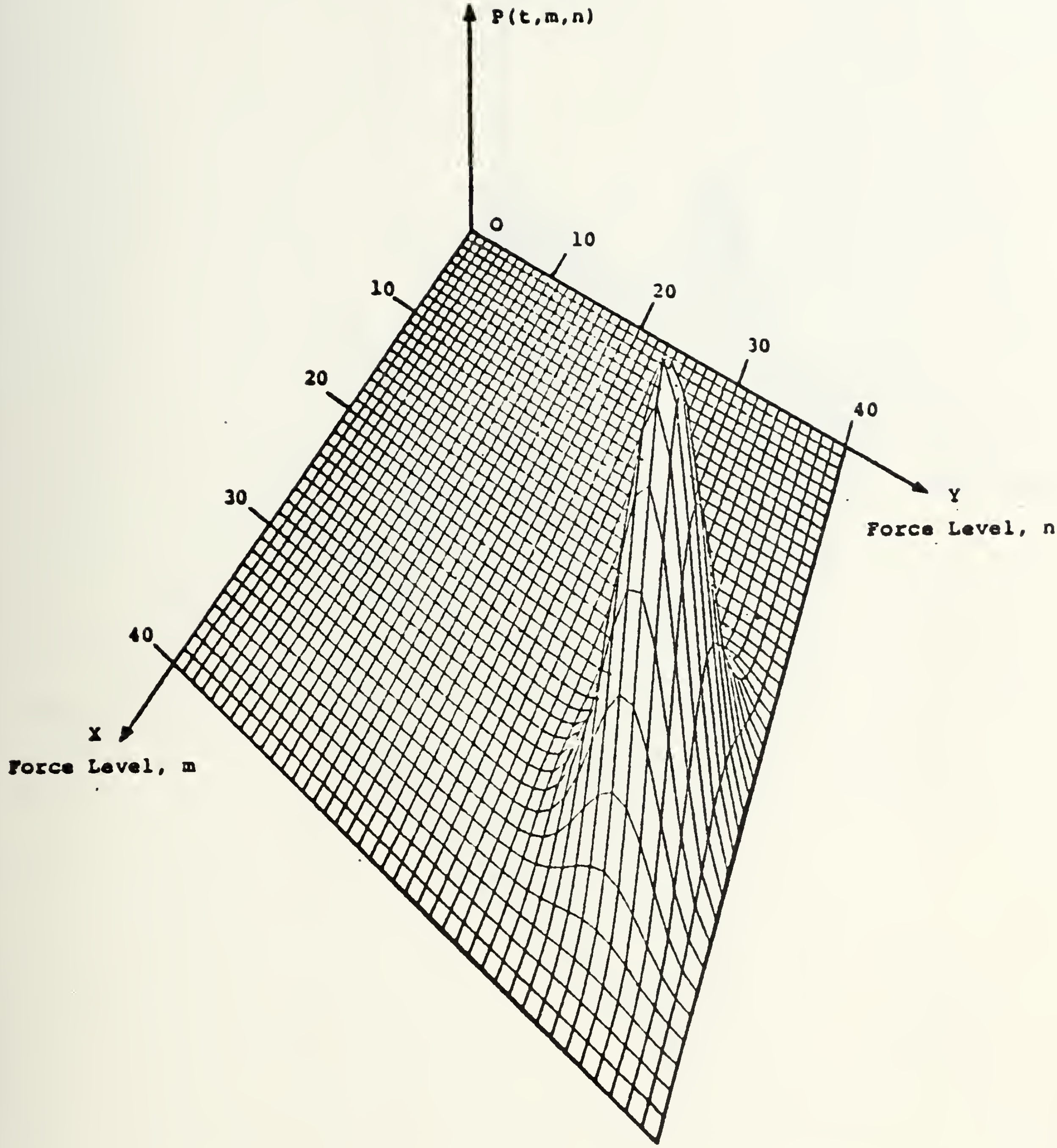

Figure 4 - PLOT OF JOINT PROBABILITIES P(t,m,n)

with data according to Table 1

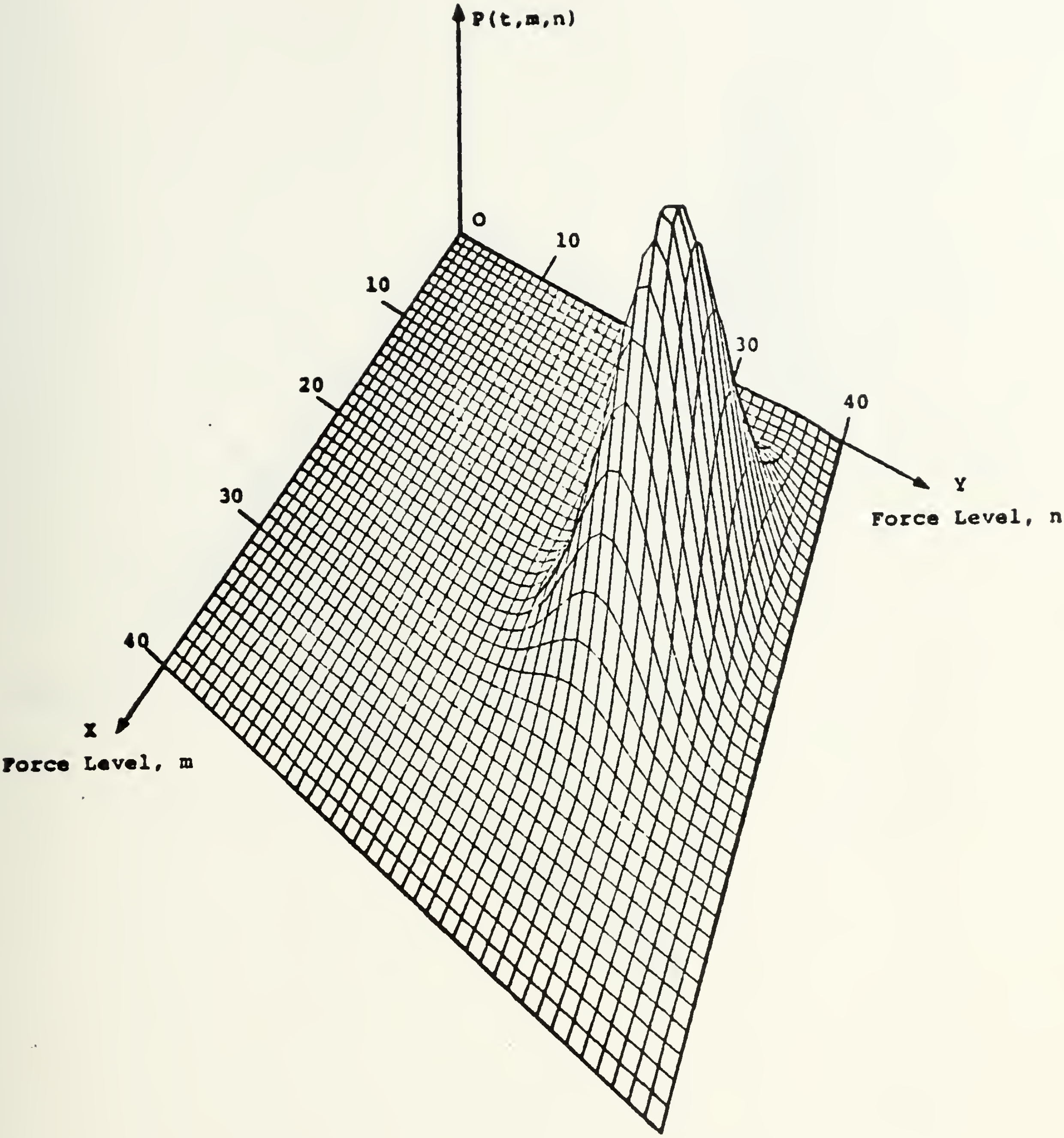

Figure 5 - PLOT OF JOINT PROBABILITIES P(t,m,n)

with data according to Table 1

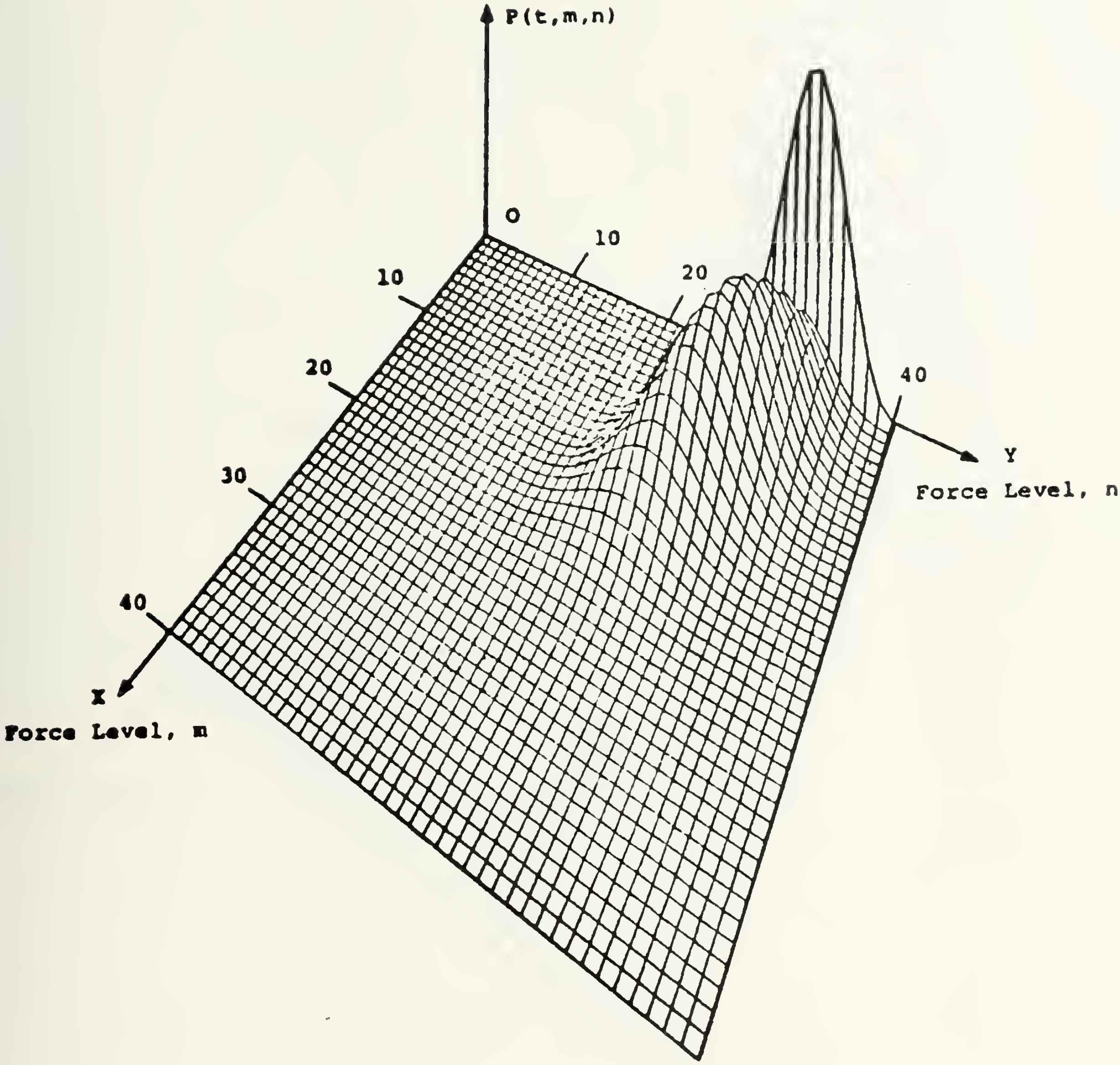

Figure 6 - PLOT OF JOINT PROBABILITIES P(t,m,n)

with data according to Table 1

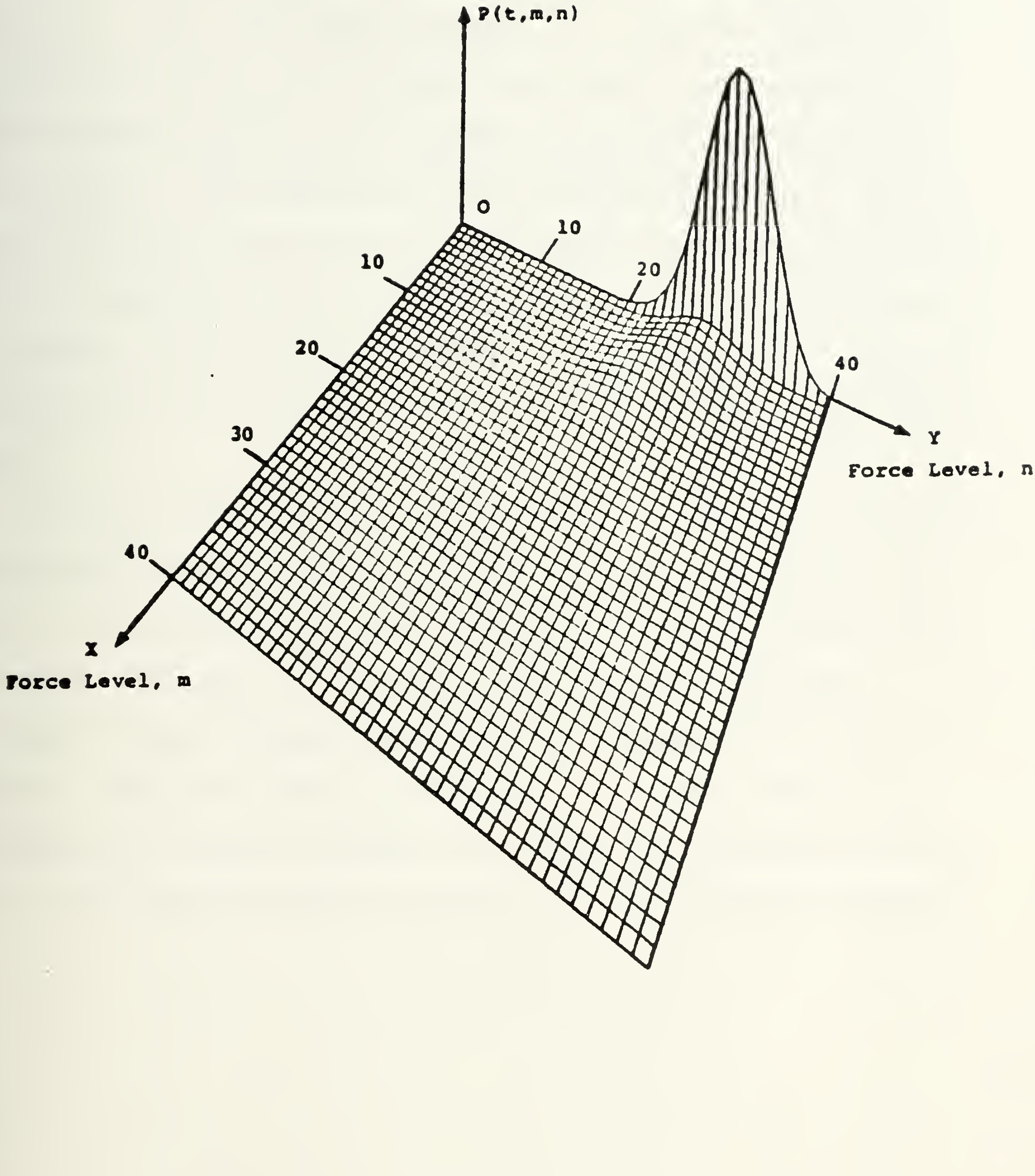

Figure 7 - PLOT OF JOINT PROBABILITIES P(t,m,n)

with data according to Table 1

At the time $t = 0$ all probability is located at (m_0, n_0) in the state space (Region I). As time passes, the probability mass is distributed over more states, with the mode moving away from the diagonal towards the winning side. All points in the state space with breakpoints, i.e. (m_{bp}, n) and (m, n_{bp}) for all m and n, are absorbing states, probability mass is absorbed in that states. The sum of probability mass in Region V (see Fig. 2) represents the probability, that the N-force wins at that given time, in Region VI that the M-force wins.

The next five figures show a similar sequence of plots for the joint probability $P(t,m,n)$ for the force levels $M(t)$ and $N(t)$. The data is explained in Table 2. Note the small differences because of the nonzero force level breakpoints. Probability mass having reached the breakpoint "piles" up there. The state space is reduced by the fixed force level breakpoints, but the probability distribution evolves basically in the same qualitative manner as in the previous example.

TABLE 2

Data for the Numerical Example 2

Force Levels	$m_0 = 40$	$m_{bp} = 8$
	$n_0 = 40$	$n_{bp} = 8$
Attrition Rates	$a = 0.008$	M casualties per minute and N firer
	$b = 0.004$	N casualties per minute and M firer
At Times	$t_1 = 0.025\ t_f$	
	$t_2 = 0.250\ t_f$	
	$t_3 = 0.500\ t_f$	
	$t_4 = 0.750\ t_f$	
	$t_5 = 1.000\ t_f$	

where $t_f = 120.68$ minutes is the time a deterministic battle with the same force levels and unit breakpoints ends, i.e. $x(t_f) = 8.0$ and $y(t_f) = 28.83$

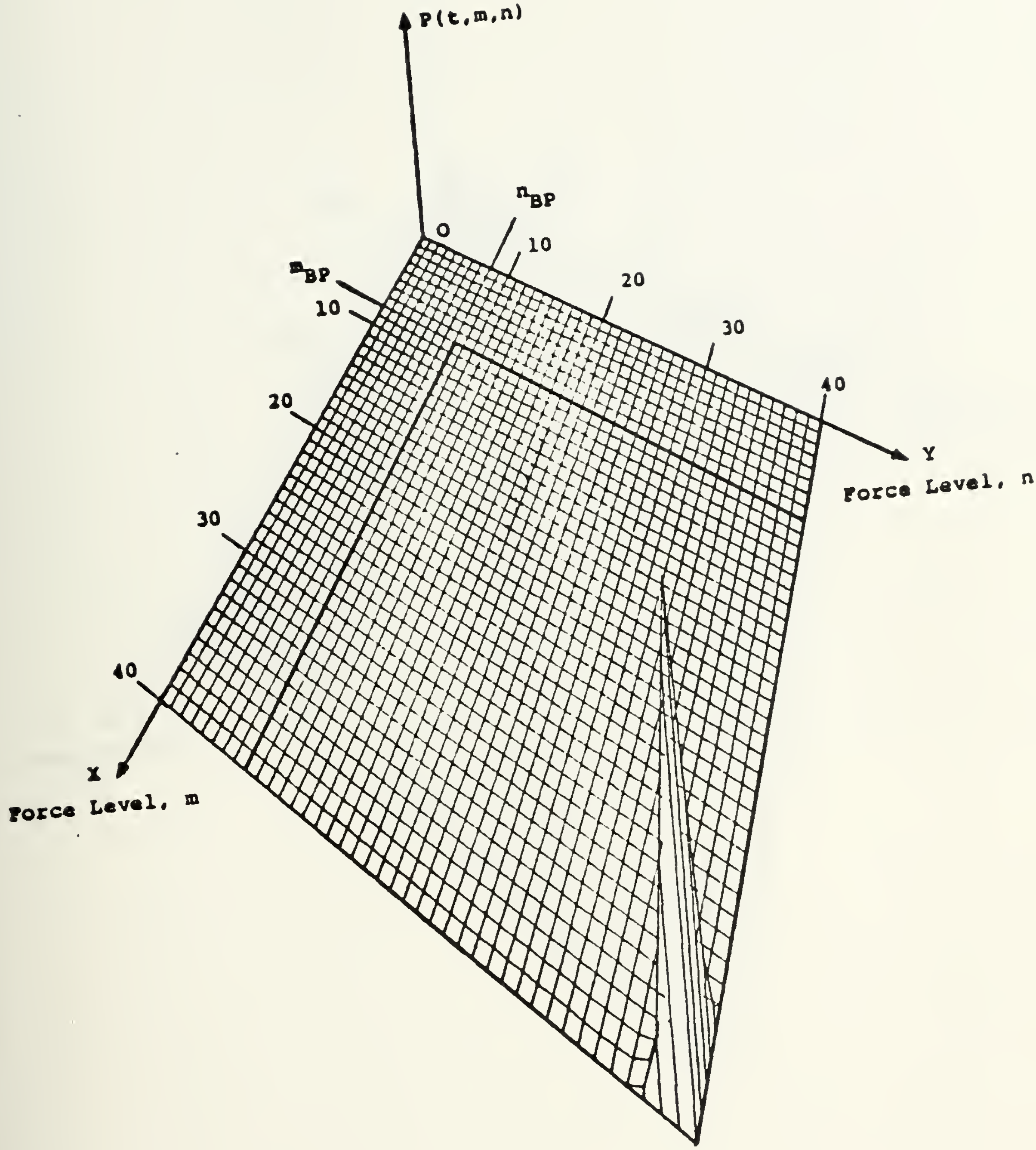

Figure 8 - PLOT OF JOINT PROBABILITIES P(t,m,n)
with data according to Table 2

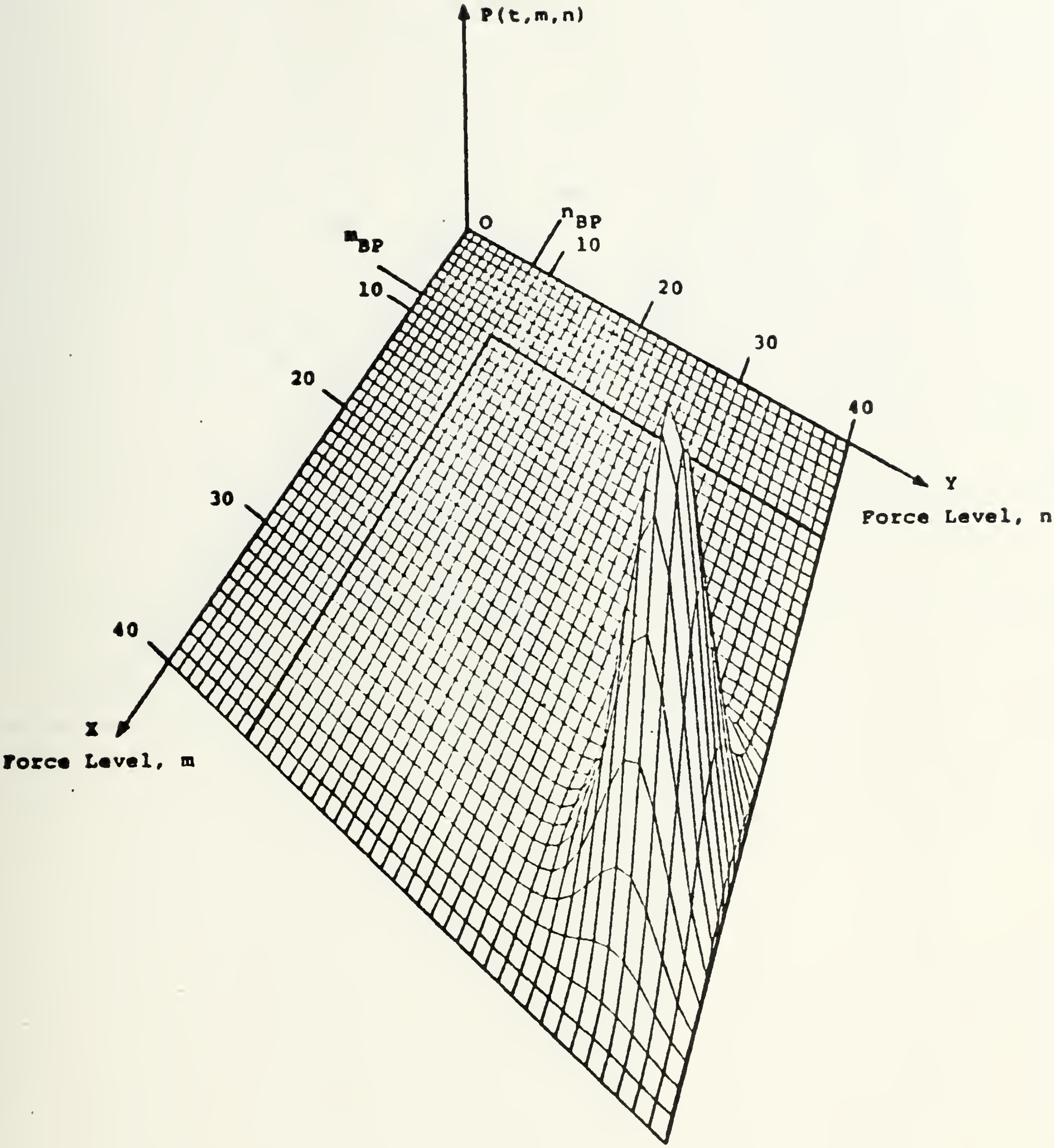

Figure 9 - PLOT OF JOINT PROBABILITIES P(t,m,n)

with data according to Table 2

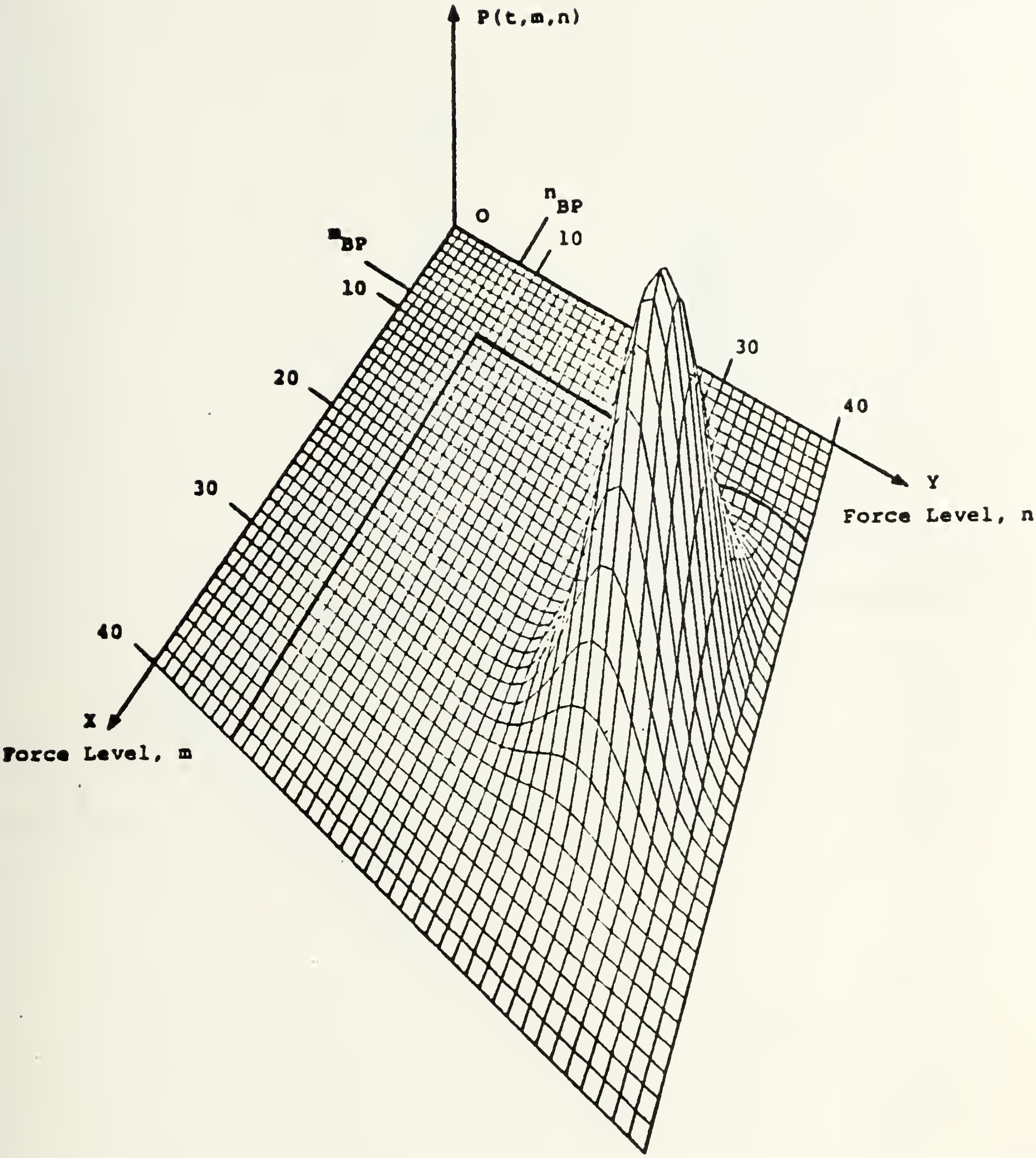

Figure 10 - PLOT OF JOINT PROBABILITIES P(t,m,n) with data according to Table 2

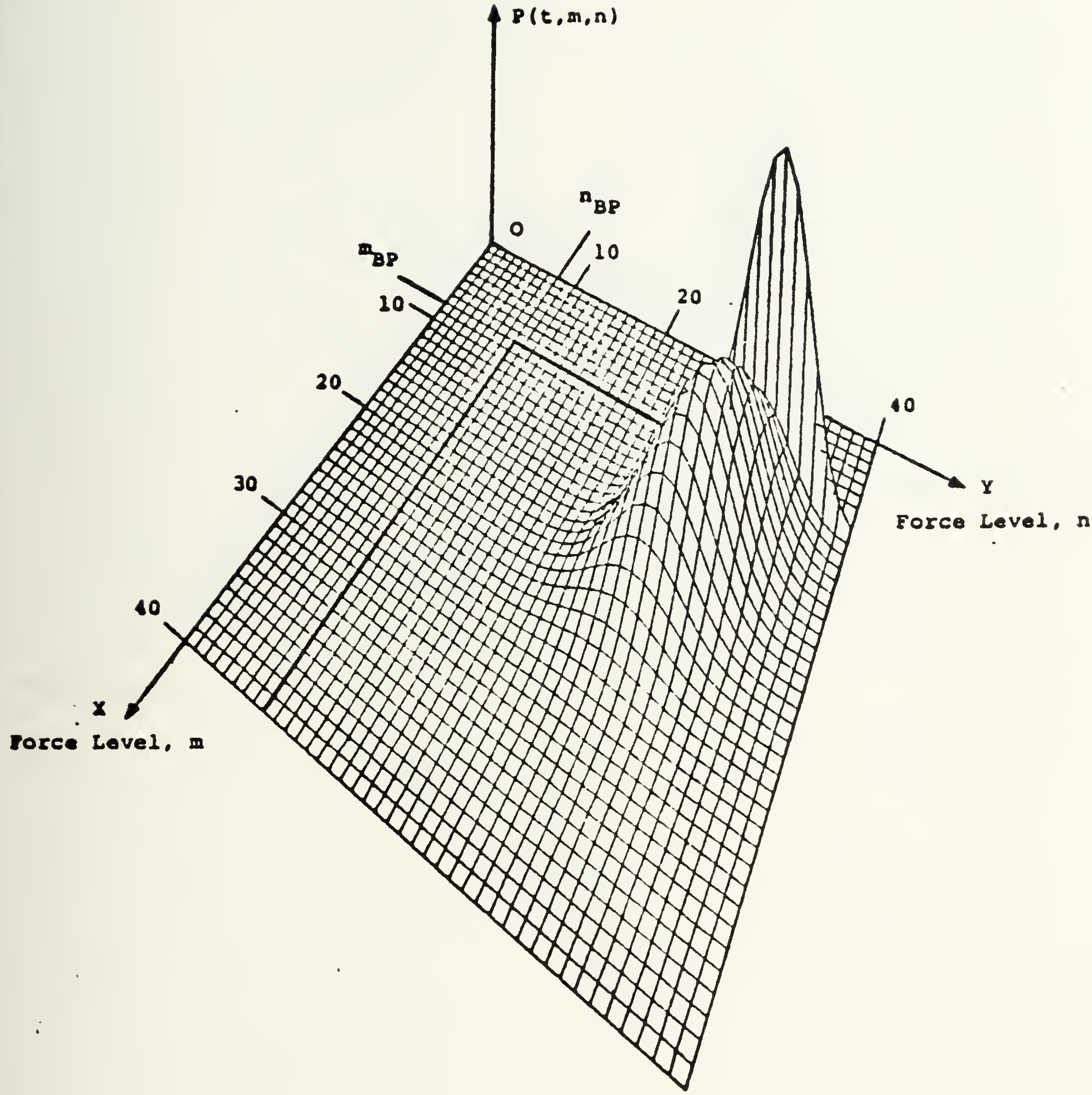

Figure 11 - PLOT OF JOINT PROBABILITIES P(t,m,n)

with data according to Table 2

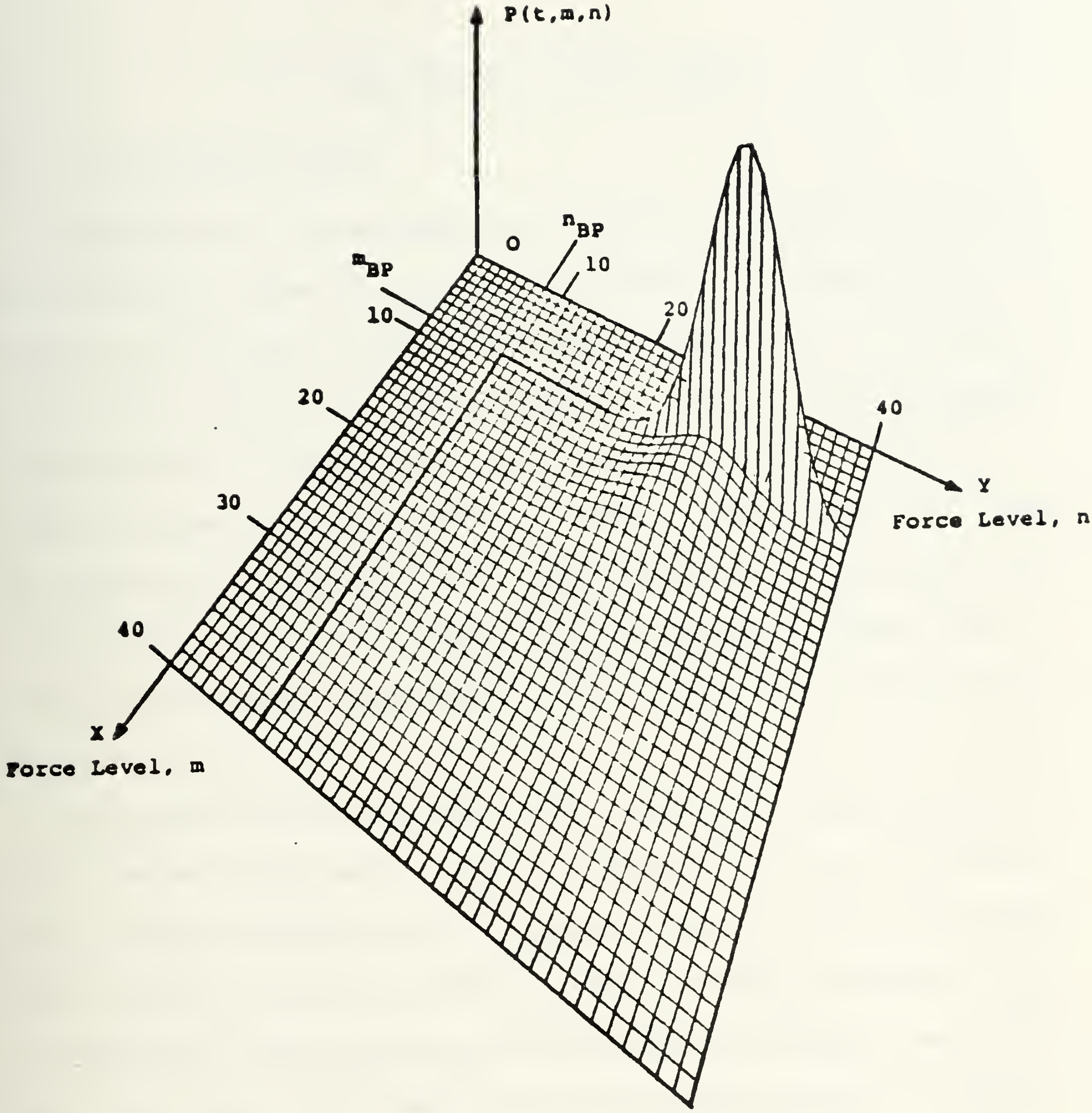

Figure 12 - PLOT OF JOINT PROBABILITIES P(t,m,n)

with data according to Table 2

IV. EVOLUTION OF THE FORCE LEVELS

A. THE DETERMINISTIC MODEL

The evolution of force levels as a function of time, X(t) and Y(t), has already been stated in equations (2.5) and (2.6). The next two figures show the force levels with different breakpoints. It is easy to realize that introducing a nonzero breakpoint does not change the underlying function. In other words, using X(t) as an example, for both figures the same curve was used but at the point where the X-force reaches its breakpoint, the curve is "cut." From that point in time, there are no more changes in the force levels. So introducting a nonzero breakpoint only shifts the discontinuity (marked by DX1 in Fig. 13) up along the curve to DX2.

The probability for one side to win is either one or zero, because it is a deterministic model. To easily determine which side is going to win, a victory prediction condition can be obtained by solving each force level equation (2.5) and (2.6) for the time to reach its breakpoint tx_{bp} by substituting $X(t)=x_{bp}$ and ty_{bp} by substituting $Y(t)=y_{bp}$. Then X will win if $ty_{bp} < tx_{bp}$, which leads to the prediction condition. X will win a fixed force breakpoint battle if and only if

$$x_0/y_0 > \frac{a(y_0^2 - y_{bp}^2)}{b(x_0^2 - x_{bp}^2)} \quad . \qquad (4.1)$$

This shows that given the initial data one can predict the outcome of the battle in terms of force levels and time until the battle finishes.

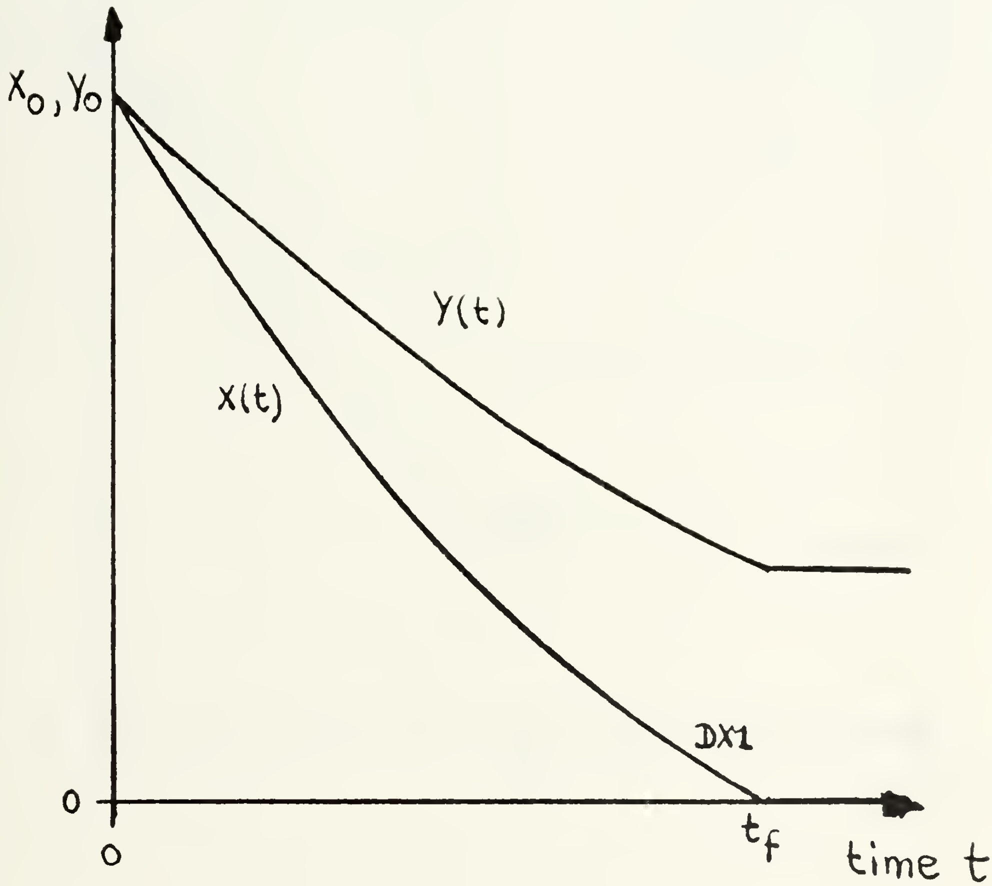

Figure 13 - FORCE LEVELS OVER TIME

Deterministic Model with Breakpoint $x_{bp} = 0$

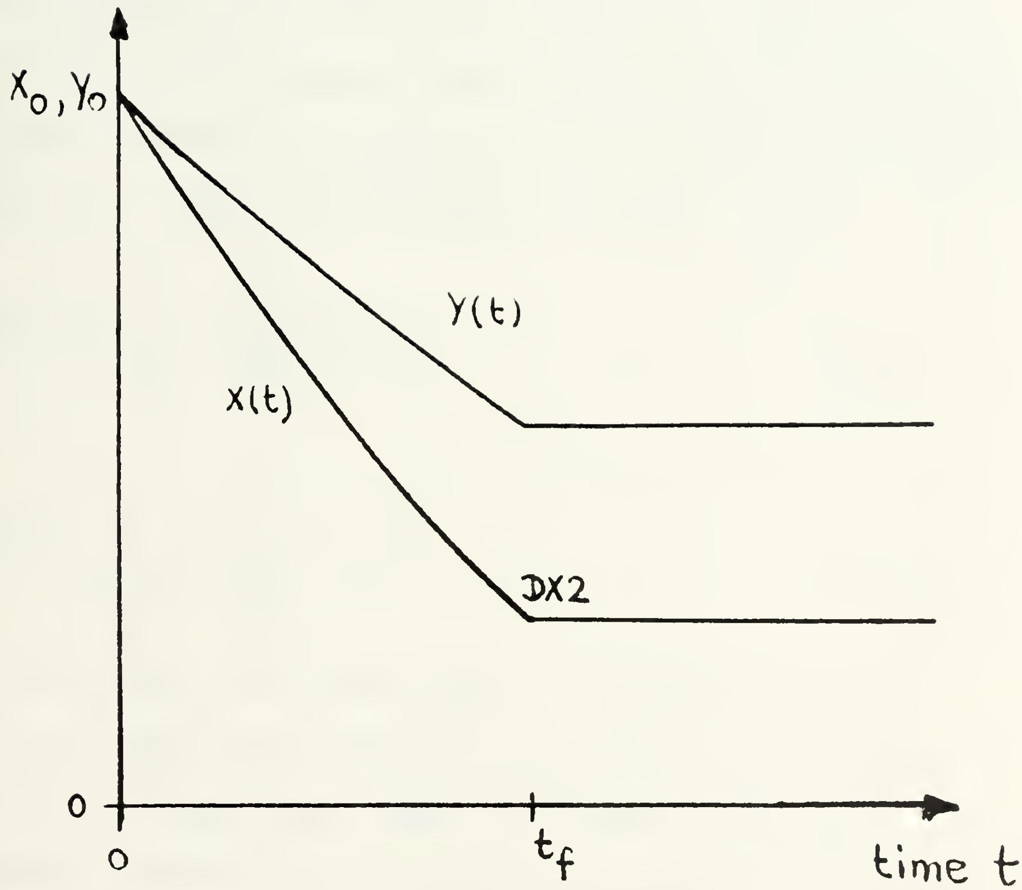

Figure 14 - FORCE LEVELS OVER TIME

Deterministic Model with Breakpoint $x_{bp} > 0$

B. THE STOCHASTIC MODEL

Every possible state (m,n) in the state space has associated with it a certain probability between zero and one which is a function of time. In order to gain more insight into the stochastic process not a single realization of a battle has to be considered, but an average battle. Therefore the expected value of the force levels (i.e. averages) as a function of time and the variances in the force levels were investigated.

The straight forward way to compute the expected force levels involves the knowledge of the probability distribution P(t,m,n). Then

$$E(M(t)) = \sum_{m_{bp}}^{m_0} m \sum_{n_{bp}}^{n_0} P(t,m,n) \qquad (4.2)$$

$$E(N(t)) = \sum_{n_{bp}}^{n_0} n \sum_{m_{bp}}^{m_0} P(t,m,n) \qquad (4.3)$$

are the expected values of the force levels as a function of time.

There are some other ways to calculate the expected force levels, one of which will be stated here. Recall the "hybrid-analytical-numerical" method to get the state probabilities (equations 3.5 through 3.15). CLARK [4] has also shown that the i^{th} moment of, for example, the M-force level may be computed as

$$E(M^i(t)) = D_{0,0}^{(i)} + \sum_{j=1}^{m_0} \sum_{k=1}^{n_0} D_{j,k}^{(i)} \exp(-(ak+bj)t) \qquad (4.4)$$

with for $1 \le j \le m_0$ and $1 \le k \le n_0$

$$D^{(i)}_{j,k} = \sum_{m=1}^{j} m^i \sum_{n=0}^{k} C^{m,n}_{j,k} \quad , \qquad (4.5)$$

$$D^{(i)}_{0,0} = \sum_{m=1}^{m_0} m^i \quad C^{0,n}_{0,0} \quad . \qquad (4.6)$$

This again emphasizes the strong point of CLARK's solution. For a given set of battle parameters the coefficients $C^{m,n}_{j,k}$ have to be computed only once. Then with this information and relatively small computational effort not only the state probabilities but also the first and second moment of the force levels can be computed. This determines the variance in the force levels at the same time, e.g.

$$Var(M(t)) = E(M^2(t)) - E(M(t)) \cdot E(M(t)) \quad . \qquad (4.7)$$

On the other side, the weak point is that CLARK [4] considered only breakpoints equal zero.

Many authors have discussed one side's probability of winning or probability of winning conditioned on a certain number of survivors, which always eliminated the parameter time by integrating from time t=0 to infinity. This may be legitimate to answer absolute (meaning time independent) questions about who will win, but for direct comparisons with the deterministic model, this author has the feeling that the best question to ask regarding a winner is:

What is the probability of one side winning given the stochastic battle lasted as long as the deterministic one?

The calculation of these probabilities gives another interesting probability, because given the time $t = t_f$ (final time of the deterministic battle)

P(battle has not yet finished) =

$$1-P(M \text{ wins} | t=t_f)-P(N \text{ wins} | t=t_f) \quad . \tag{4.8}$$

There has also been work done regarding the distribution of the time to finish a battle. But this is beyond the scope of this thesis (SPRINGALL [9]).

C. DIFFERENCES IN THE FORCE LEVELS

The deterministic model, especially the force level equations (2.5) and (2.6), describe a process with continuous state parameters where, in reality, the possible states are integers. Quoting from LANCHESTER [7]:

> Since the forces actually consist of a finite number of finite units (instead of an infinite number of infinitesimal units) the end of the curve must show discontinuity, and break off abruptly when the last man is reached; the law based on averages evidently does not hold rigidly when the numbers become small.

LANCHESTER suggested that his differential equations may be good approximations only as long as the force sizes are large. He also stated that the equations are based on averages, implying an underlying stochastic process.

This shows that there must be a difference in the force levels which should become significant when the number of combatants is small. This difference was called <u>bias</u> by CLARK [4] and TAYLOR [10]. It can be shown that

$$\frac{dE(M(t))}{dt} = -aE(N(t))+aB_n(t) \tag{4.9}$$

and

$$\frac{dE(N(t))}{dt} = -bE(M(t))+bB_m(t) \tag{4.10}$$

where

$$B_n(t) = n_{bp} \sum_{m_{bp}+1}^{m_0} P(t,m,n_{bp}) + \sum_{n_{bp}+1}^{n_0} nP(t,m_{bp},n) \qquad (4.11)$$

and

$$B_m(t) = m_{bp} \sum_{n_{bp}+1}^{n_0} P(t,m_{bp},n) + \sum_{m_{bp}+1}^{m_0} mP(t,m,n_{bp}) \; . \qquad (4.12)$$

The bias terms $B_m(t)$ and $B_n(t)$ can be interpreted as the expected values of M(t) or N(t) conditioned on the fact that the battle has already ended at time t, for example,

$$B_m(t)=E\left(M(t)) \mid N(t)=n_{bp} \text{ or } M(t)=m_{bp}\right) \qquad (4.13)$$

In other words, equation (4.9) says the expected casualty rate of the M-force is proportional to the expected number of survivors of the N-force given neither of the two forces has reached its breakpoint.

Define the bias of the X-force as $\Delta x(t) = E(M(t))-X(t)$ and the bias of the Y-force as $\Delta y(t) = E(N(t))-Y(t)$. Then using equations (2.8) and (2.9) together with (4.9) and (4.10) it follows that

$$\frac{d}{dt}\Delta x = -a\Delta y+aB_n(t) \qquad (4.14)$$

$$\frac{d}{dt}\Delta y = -b\Delta x+bB_m(t) \qquad (4.15)$$

with the initial conditions x(0)=) and Y(0)=). This has the solution

$$\Delta x(t)=\sqrt{ab} \int_0^t \{B_n(s)\sqrt{a/b}\cosh(\sqrt{ab}\,(t-s)$$

$$-B_m(s)\sinh(\sqrt{ab}\,(t-s))\}ds, \qquad (4.16)$$

and

$$\Delta y(t)=\sqrt{ab}\int_0^t \{B_m(s)\sqrt{b/a}\cosh(\sqrt{ab}(t-s)) - B_n(s)\sinh(\sqrt{ab}(t-s))\}ds \ . \qquad (4.17)$$

Since for a fixed, nonnegative argument z, the cosh(z) is always greater than the sinh(z), it is easy to visualize that in most of the cases both biases are positive, meaning the expected force levels of the stochastic model are higher than the deterministic force levels. This has been shown by CLARK [4] and CRAIG [5] and confirmed by this author. In the rest of the cases the winner's bias is negative or close to zero and the loser's bias is positive. Two examples are given in Table 3.

TABLE 3

Examples for cases where one bias is positive and the other bias is negative.

A. Y-force wins in a fight to the finish

$x_0 = y_0 = 40$	$a = 0.08$
$x_{bp} = 0$	$b = 0.04$
$y_{bp} = 0$	
$t_f = 15.581$	$\Delta x(t_f) = 3.22$
	$\Delta y(t_f) = -0.23$

B. Y-force wins in a fight with equal initial force levels, but different breakpoints.

$x_0 = y_0 = 15$	$a = 0.08$
$x_{bp} = 12$	$b = 0.08$
$y_{bp} = 6$	
$t_f = 2.85$	$\Delta x(t_f) = 0.15$
	$\Delta y(t_f) = -0.10$

An interesting point has to be mentioned regarding case B of Table 3. Here the expected force level of the winner is smaller than the expected force level of the loser at the time a deterministic battle ends.

To get a better feeling for the differences in the force levels, the next two figures show as an example a large spectrum of force level behavior. The data and notation is described in Table 4. There are four battles with four different breakpoints drawn as they evolve over time from zero to the time an equivalent deterministic battle ends.

TABLE 4

$X_0 = 40$	$X_{bp}(i) = 40-0.2i$
$Y_0 = 40$	$Y_{bp}(i) = 40-0.2i$
$a = 0.09$	for $i = 1,2,3,4$
$b = 0.07$	

$Mi = E\ M(t)$

$Ni = E\ N(t)$

for $i = 1,2,3,4$

Where the index i corresponds to the battle with the i^{th} breakpoint force level:

X-force is always loser

Y-force is always winner

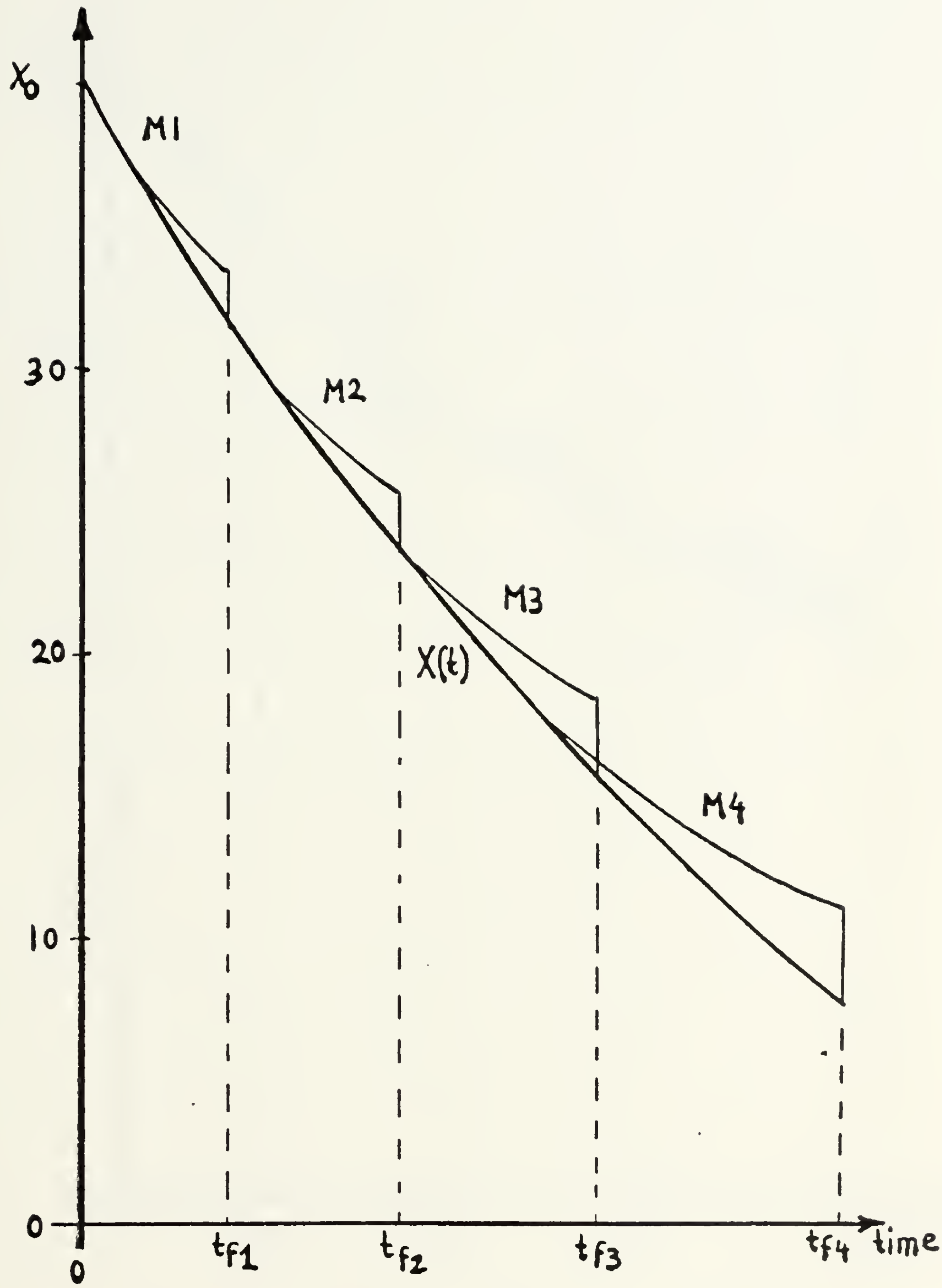

Figure 15 - DETERMINISTIC AND EXPECTED FORCE LEVELS FOR THE X - FORCE

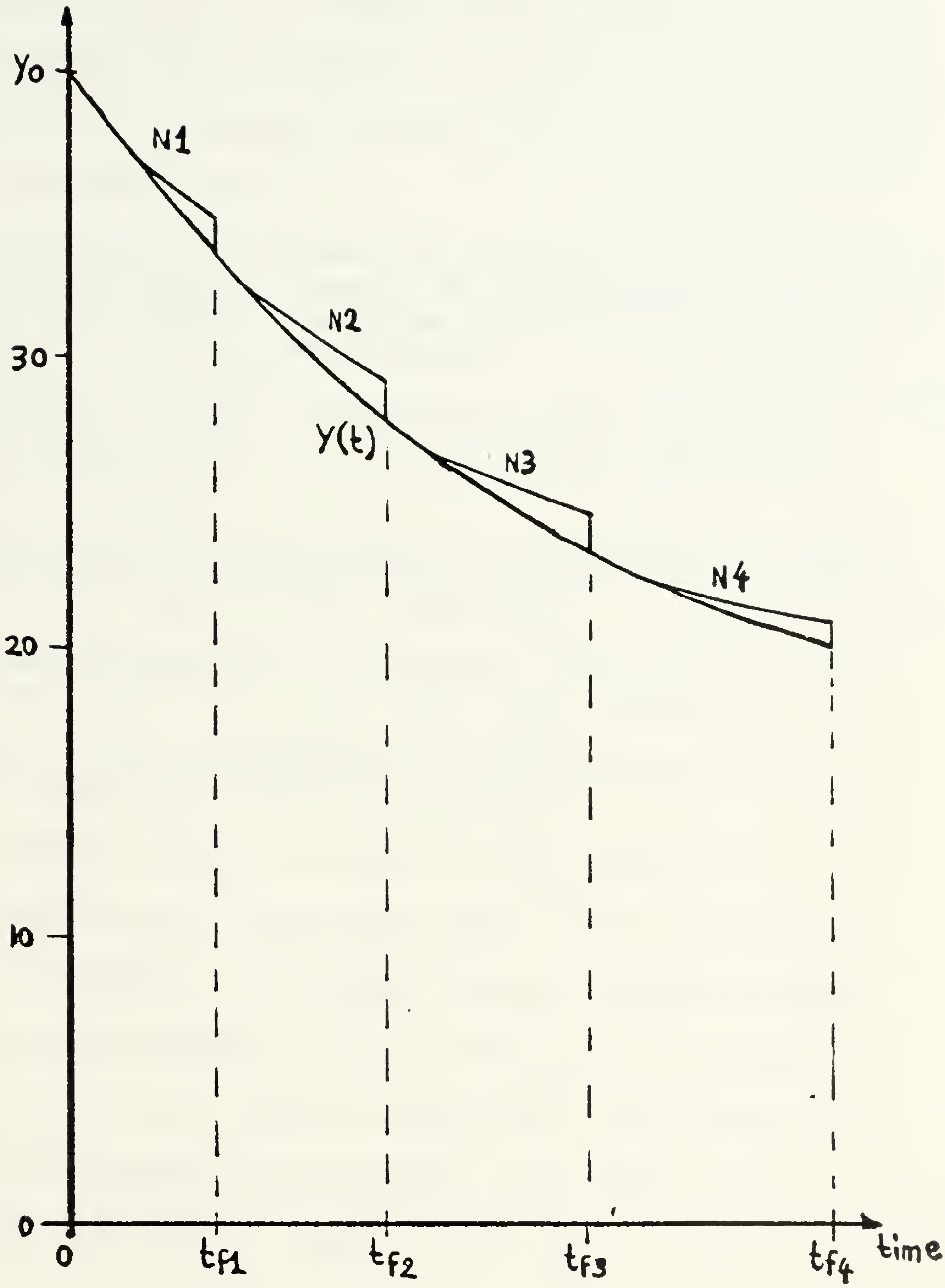

Figure 16 - DETERMINISTIC AND EXPECTED FORCE LEVELS FOR THE Y - FORCE

CRAIG [5] has formulated hypotheses concerning the biases in the average force levels based on his work. His hypotheses were partially confirmed, but in some cases they have to be modified. Therefore, another similar set of hypotheses will be given and supported by Fig. 15 and Fig. 16, as well as some of the later figures.

H 1) Given fixed initial force levels and attrition rate coefficients, as the breakpoint force levels increase, the numerical bias for the loser decreases. The biases for the winner do not show this monotone trend except for the case of symmetric parity.

H 2) Everything else constant, the bias of the loser increases with increasing initial force levels; this is also true in the symmetric parity case for both forces.

H 3) Given the initial force level ratio is close to one at the time corresponding to the end of the deterministic battle, the bias of the loser is always larger than the bias of the winner.

H 4) At the time corresponding to the end of a deterministic battle, the biases become larger as the forces come closer to parity.

H5) The biases at times corresponding to less than one half the duration of the deterministic battle are negligible.

The case of symmetric parity, i.e. equal initial force levels, breakpoints and equal attrition rate coefficients seems to be kind of a "limiting" case. For example, at parity the biases of both forces behave in the same manner and are equal. In Fig. 17 the biases at the time a deterministic battle ends as a function of the initial force levels and as a function of the breakpoints are presented. It is also another verification for the hypotheses H 1 and H 2.

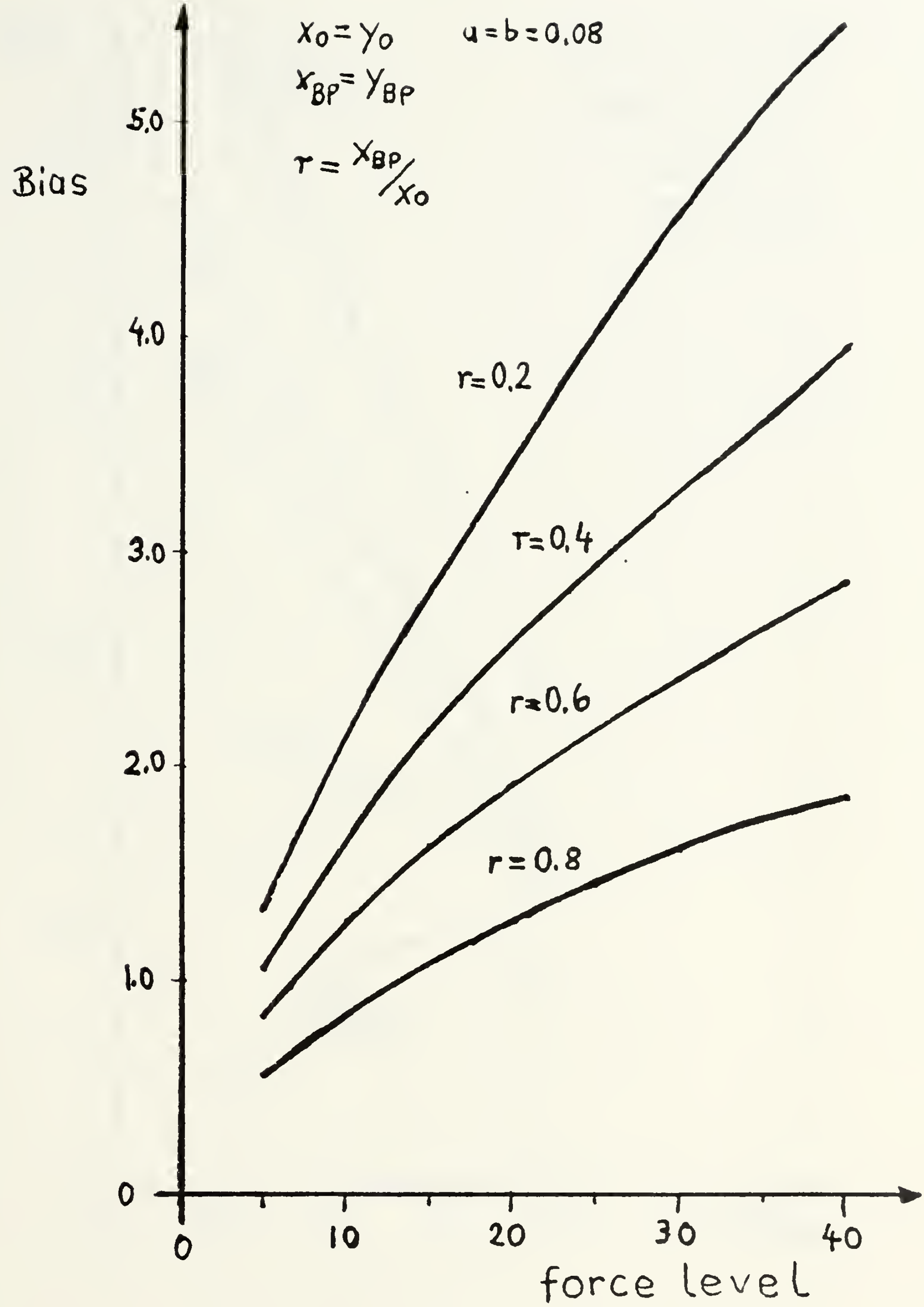

Figure 17 - BIASES IN SYMMETRIC PARITY

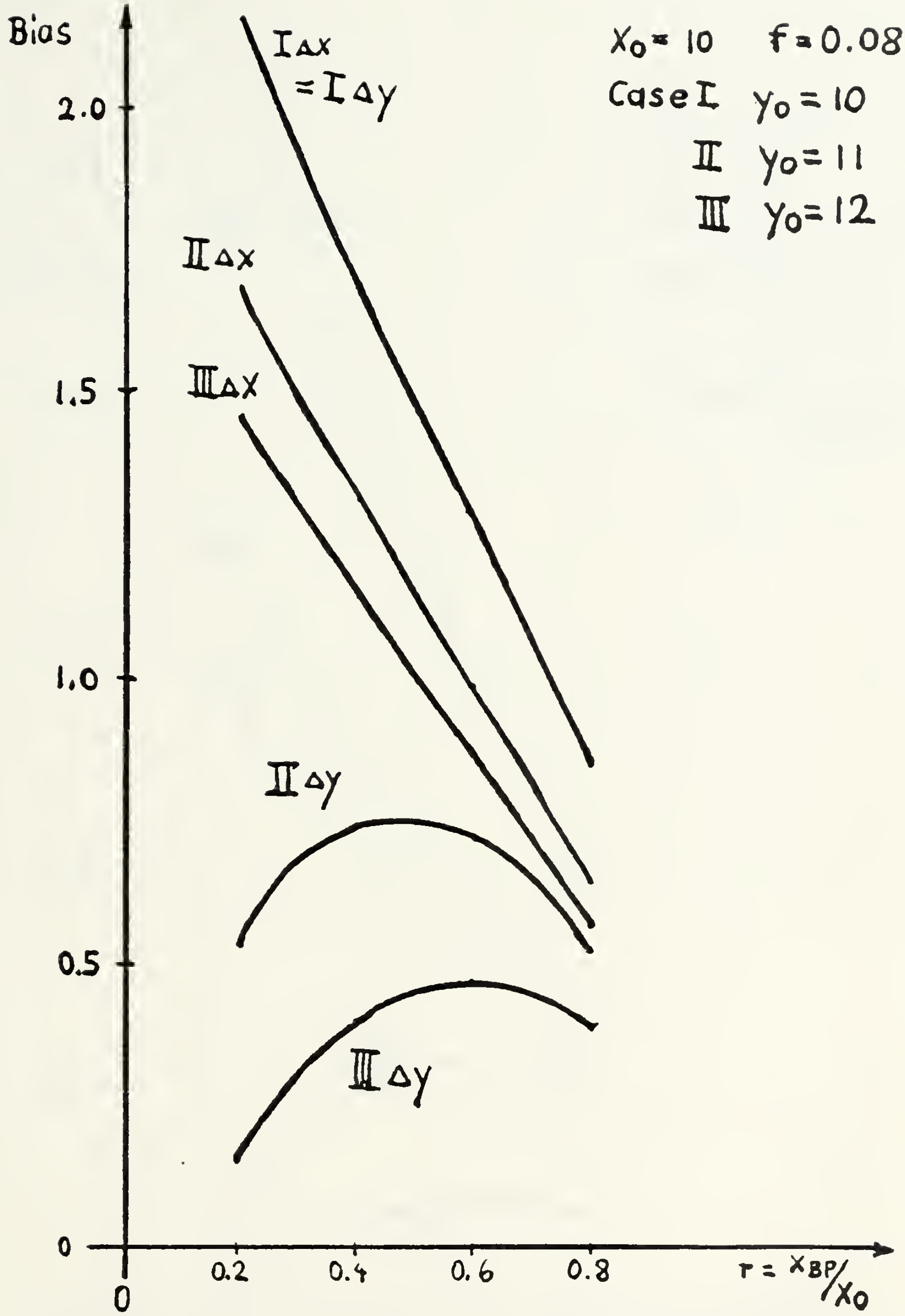

Figure 18 - BIASES WITH DIFFERENT INITIAL FORCE LEVELS

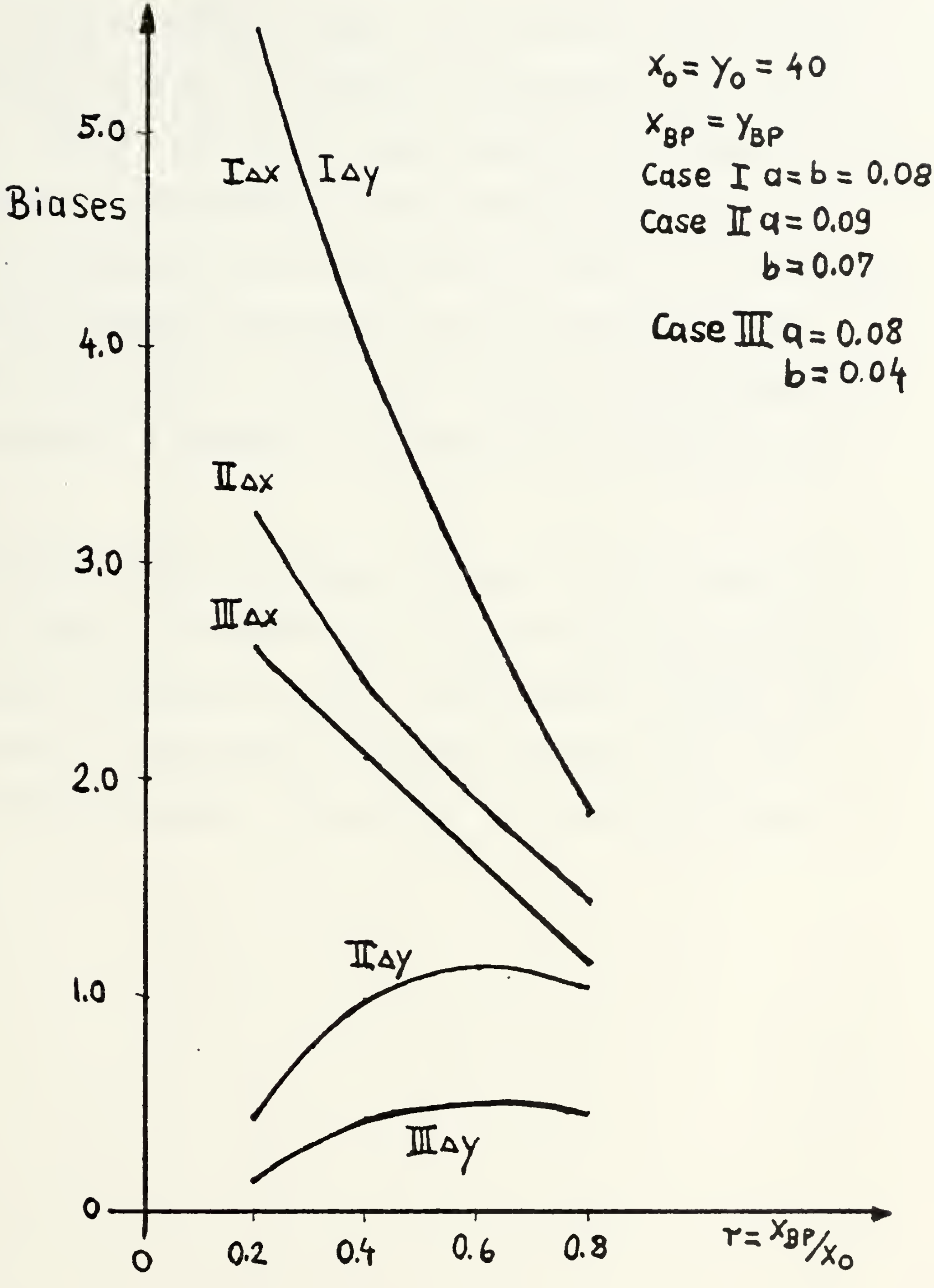

Figure 19 - BIASES WITH DIFFERENT ATTRITION RATE COEFFICIENTS

Considering the changes in magnitude of the biases for battles like in Fig. 15, 16 and 17, CRAIG [5] came to the conclusion that when the forces are closer to parity, the biases at the deterministic battle's end increase. Several battles were fought starting with symmetric parity and then varying the force levels and the attrition rate coefficients. Sample results are shown in Fig. 18 and Fig. 19. In Fig. 18, the initial force levels were changed giving the Y-force ten and twenty percent higher initial force levels. The biases at the end of the deterministic battle are plotted as a function of the breakpoint force level ratio $r = x_{bp}/x_0$.

Fig. 18 supports CRAIG's hypothesis, as does Fig. 19. Here not the initial force levels but the attrition rate coefficients were changed in order to deviate from symmetric parity. The last way to deviate from symmetric parity is a case where the deterministic model gives equal answers for different battles. The data and the results are shown in Table 5.

TABLE 5

Equal initial force level battle with non equal breakpoints

$X_0 = 15$ $\quad Y_0 = 15$ $\quad a = b = f = 0.08$

X_{bp}	Y_{bp}	$\Delta x(t_f)$	$\Delta y(t_f)$
12	12	1.08	1.08
12	9	0.71	0.63
12	6	0.15	-0.10
12	3	0.68	0.58
9	9	1.62	1.62
9	6	1.14	0.93
9	3	1.06	0.75
6	6	2.15	2.15
6	3	1.67	1.22
3	3	2.84	2.84

The X-force level at time t_f is the same as the Y-force level. Thus, the expected force level for the winner (i.e. the Y-force) is smaller than the expected force level of the loser (the X-force), because $\Delta x(t_f)$ is always larger than $\Delta y(t_f)$. This is easy to see when one remembers the way the expected force level is computed (equation (4.3)). Since the Y-force has a lower breakpoint, there are states (m,n) possible where $n_{bp}<n<m_{bp}$. Apparently these states have a nonzero probability associated with them, which decreases the expected force level below the expected value for the X-force. This might be a starting point for further studies.

D. VARIABILITY IN THE FORCE LEVELS

Naturally in the deterministic case there does not exist any variability in the force levels. On the other hand, for the stochastic model, the variance in the force levels as a function of time is a measure of the dispersion of the number of survivors about their mean value.

CLARK [4] has hypothesized two different types of behavior for the variance in the force levels, which are shown in Fig. 20 for the data presented in Table 2. The first type of behavior is that of the variance for the N-force, i.e. the variance increases monotonely as a function of time and is asymptotic to a limiting value. It was found that this type of behavior occurs when the side is going to win and for the case of symmetric parity. The second type of behavior shown is the variance of the M-force, Var(M(t)), as a function of time. This increases to a maximum value then decreases asymptotically to a limiting value. This type of behavior is associated with the loser of the battle.

The variance in the force levels is a function of the initial and breakpoint force levels, the battle time and the attrition rate coefficients. Unfortunately one does not know what this dependence is. Based on many numerical results a set of hypotheses will be stated and the next figures will support them.

H 1) Given fixed initial force levels and attrition rate coefficients, as the breakpoint force levels increase the variance of the force levels decrease.

H 2) Everything else constant the variance of the loser's force level increases with increasing initial force levels. This is also true for both variances in the case of symmetric parity.

H 3) Given the initial force ratio is close to one at the time corresponding to the end of the deterministic battle the variance of the loser's force level is always smaller than the winner's variance.

H 4) At the time corresponding to the end of the deterministic battle the variance in the loser's force level increases as the forces come closer to parity. This trend is not true for the variance of the winner, except for the case of symmetric parity.

It was the intention of this author that the set of cases to illustrate the hypotheses are the same as in the illustrations (Fig. 17, 18 and 19) of the hypotheses about biases. So, Fig. 21 shows the force level variances for the different initial force levels and different breakpoints for the case of symmetric parity. In Fig. 22 the variances for a battle with equal attrition rate coefficients but varying initial force levels show that H 4 is only true for the loser. This point is emphasized by Fig. 23, where with constant and equal initial force levels the attrition rate coefficients were varied.

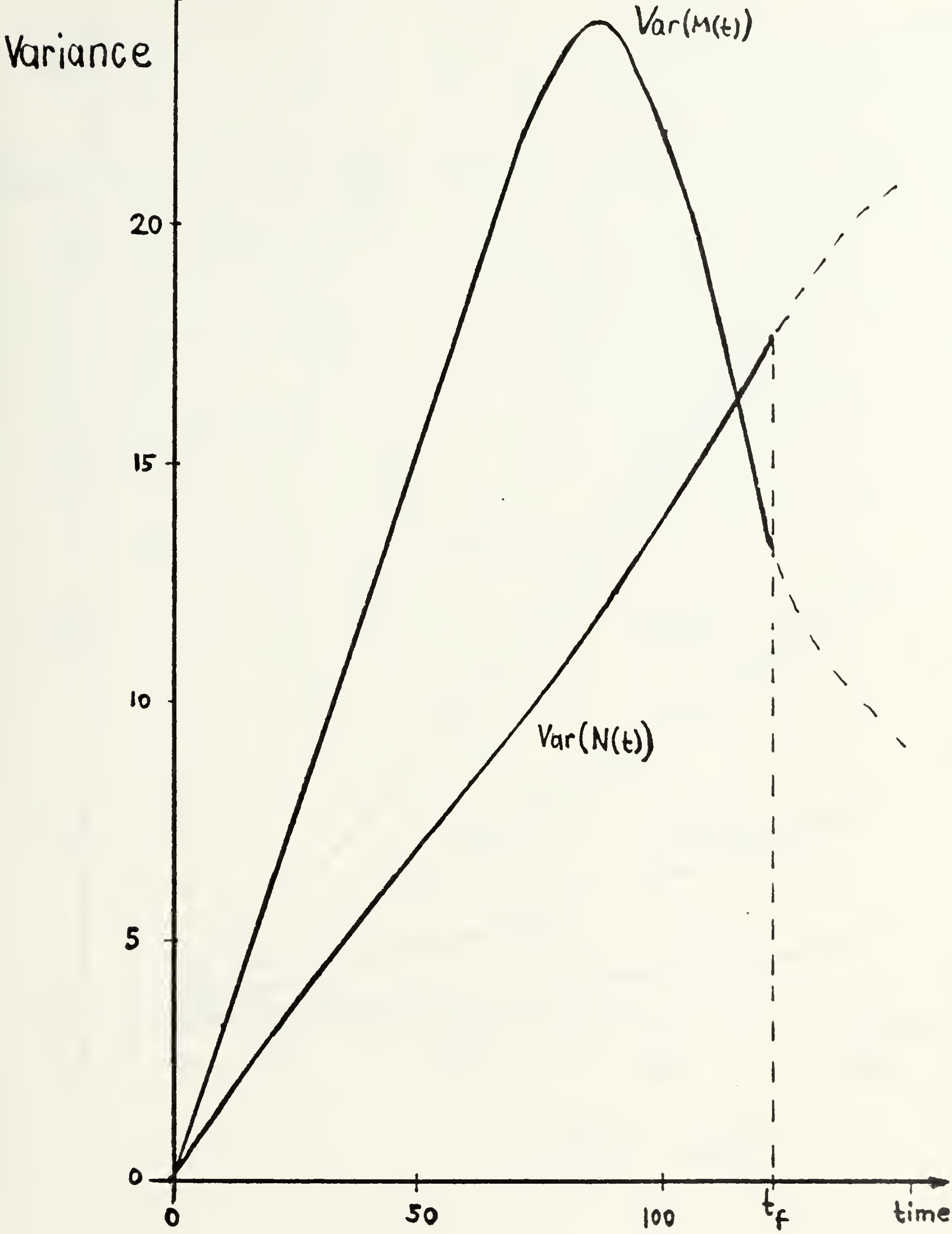

Figure 20 - FORCE LEVEL VARIANCES OVER TIME

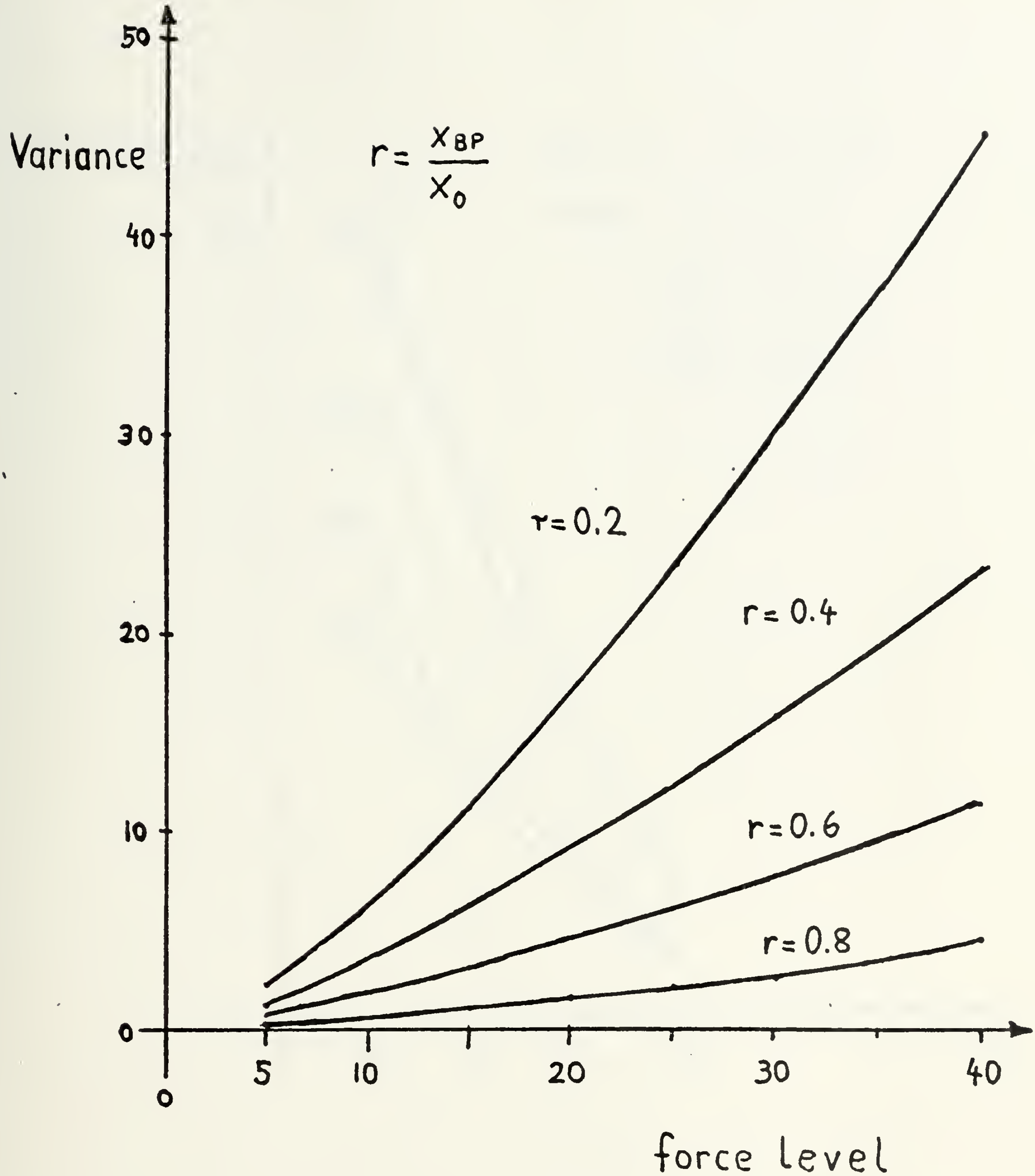

Figure 21 - VARIANCES IN SYMMETRIC PARITY

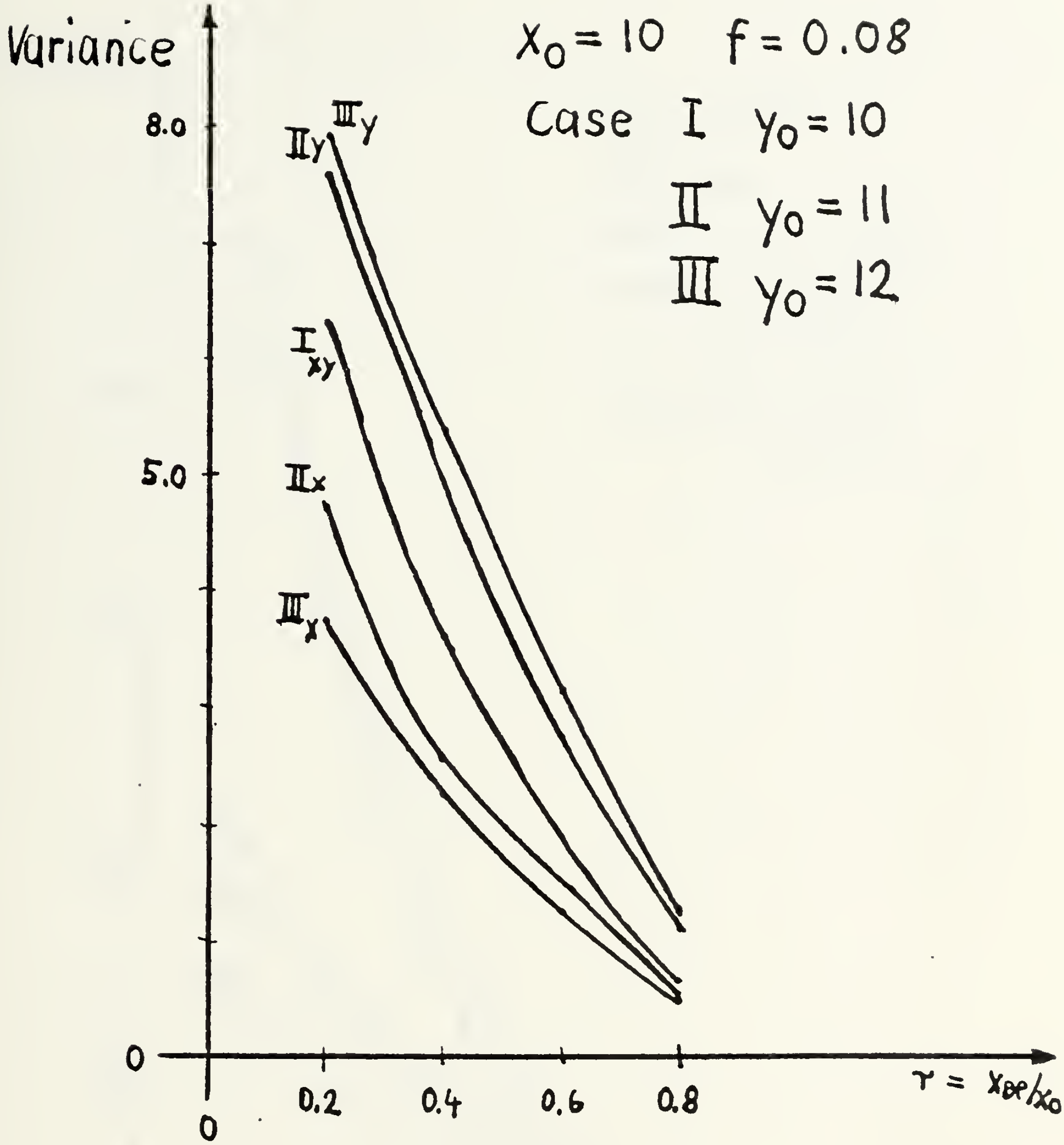

Figure 22 - VARIANCES WITH DIFFERENT INITIAL FORCE LEVELS

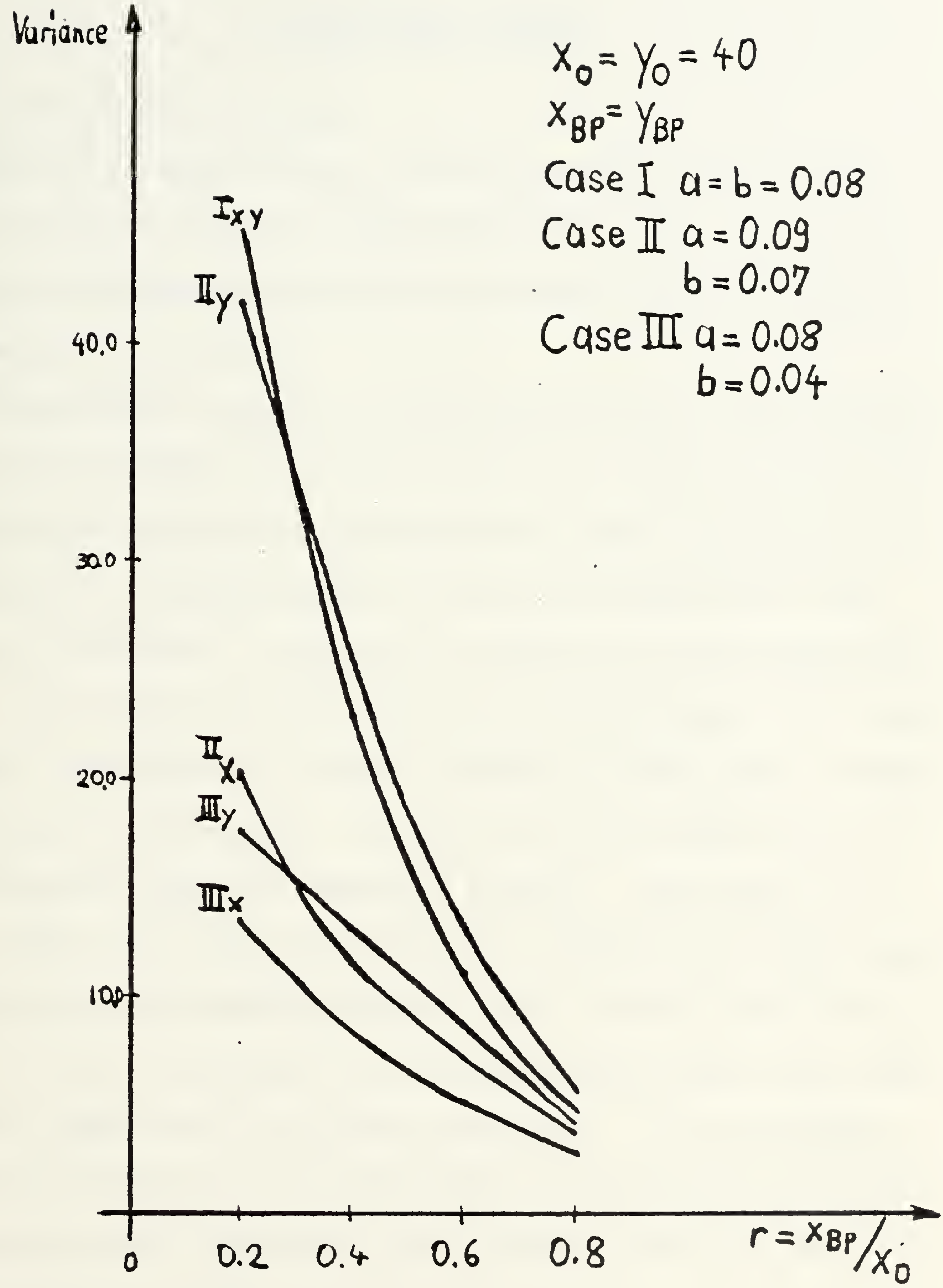

Figure 23 - VARIANCES WITH DIFFERENT ATTRITION RATE COEFFICIENTS

V. COMPUTATIONAL ASPECTS

A. USING THE NUMERICAL SOLUTION

For most of the numerical work, the state probabilities have been obtained using the fourth-order RUNGE-KUTTA method. The accuracy of the results was increased by substituting the available analytical results for the Regions I, II and III.

The disadvantage is that this solution method needs a lot of CPU-time. For the battle in Example 1, with the data given in Table 1, the calculation of the state probabilities, expected force levels and variances led to a CPU-time on an IBM-360 computer of almost 90 minutes with a time step size of 0.05 minutes. That was the reason why, in the examples the attrition rate coefficients were increased by one magnitude, which brought the computer usage down to a CPU-time of around 12 minutes with the same time step size of 0.05 minutes. The sum of the state probabilities at every time step was used as a measure of accuracy. Surprisingly, its deviation from 1.0 was always in the fourth or fifth decimal, which proves the robustness of the RUNGE-KUTTA method. Also, the fact that for the Regions I, II and III analytical solutions were substituted at each time step did not change the final outcome considerably. It was found that without the analytical partial solutions the sum of probabilities tended to be slightly higher, changing the sum of probabilities from, for example, 0.99998 to 1.00003.

B. THE EQUAL ATTRITION RATE COEFFICIENT SOLUTION

After the development of the equal attrition rate coefficient solution (EARCS) outlined in Ch. III.C, it was implemented on an IBM-360 computer. The CPU-time for the calculation of the state probabilities, expected force levels and variances for a given point in time always stayed below 20 seconds, which emphasizes its computational advantage.

Further research showed two shortcomings of the EARCS, which are easily overlooked. The coefficients C(m,n) vary over a wide range starting at 1.0 and depending on the initial force levels.

For example for a battle with

$m_0 = 20$ $\quad m_{bp} = 15$

$n_0 = 40$ $\quad n_{bp} = 20$

$f = 0.08$ $\quad t_f = 1.60995$

$C(20,40) = 1.0 \quad \text{but} \quad C(16,21) = 4.9297077276 \times 10^{33}.$

Also, the binomial coefficients in equations (3.23) and 3.24) vary over a wide range starting at 1.0 to 1352078 for the above example. This shows that the capacity the computer, in terms of the number of significant digits, is able to carry limits the implementation of this solution in the present form.

Another reason why this solution is not the end of the numerical problems is the fact that in the given form, equations (3.23) and (3.24) require computation of an alternate sum consisting of terms whose factors are relatively large binomial coefficients and small numbers between zero and one. This produces truncation errors, which yield nonsensical results like negative variances and sums of probabilities greater than one.

VI. CONCLUSIONS

Throughout this thesis, only LANCHESTER-type square-law attrition with fixed initial force levels and fixed attrition rate coefficients has been considered for a deterministic and a stochastic version. The stochastic version required much more computational effort. So, given the need for an analytical model as opposed to the use of simulation, there is not much to gain from the application of a stochastic model when the force levels are large and the forces are not near parity. In these cases the deterministic version essentially produces the same results, at least in qualitative terms. For smaller force levels or forces near parity, the stochastic version may be helpful to get more information about the dynamics of combat.

Also, there exist one case where the deterministic version cannot differentiate between several different battles. That is the case of equal initial forces, equal attrition rate coefficients, but different breakpoint force levels. This also rectifies the further development of the equal-attrition-rate-coefficient-solution (EARCS).

Using an analytical model, the discussed way of introducing randomness into the model seems not to be very enlightening. Therefore, it is suggested that another way to include random effects should be explored. The author's opinion is that working with attrition rate coefficients which are random variables seems more promissing to gain insight into the dynamics of combat. Further down the line there should be some

consideration on the usage of combinations of the possibilities to include random effects as given in Chapter II.B.

APPENDIX A

```
      C   THIS PROGRAM COMPUTES THE STATE PROBABILITIES USING THE FOURTH
      C   ORDER RUNGE-KUTTA METHOD. PLOTS OF THE EXPECTED FORCE LEVELS AND
      C   VARIANCES AS WELL AS  3-D-PLOTS FOR THE STATE PROBABILITIES ARE
      C   OBTAINED.
      C
            DIMENSION P1(33,33), P2(33,33), T3D(20)
            DIMENSION XMT(500), XNT(500), QT(500), VM(500), VN(500),
           1 VMB(551), VNB(551), VQ(551), VARX(551), VARY(551), TIM(551)
            DIMENSION EST(313),D(313),F(2),SIZE(2),KX(100),KY(100),WK(41,41,3)
           1        ,               PC(41,41),CST(41),CETERM(41)
            LOGICAL*1 IDN(41,41)
            DIMENSION DETERM(313)
            REAL*8 TTL(12)/12*'        '/
            REAL K1, K2, K3, K4
            CALL ERRSET(208,600,-1,1)
            F(1)=0.
            F(2)=0.
            LINES=0
            SIZE(1)=6.
            SIZE(2)=8.
            QFLAG = 0.
            NKXY=100
            READ(5,21) MBP, MO, NBP, NO
            READ(5,22) AA, BB
            READ(5,22)  H, FTIM
            READ(5,21) N3D
            READ(5,22) EPSQ, EPSPTT
            T3D(1) = 9999.
            IF (N3D .LT. 1) GO TO 25
            READ(5,22)  (T3D(I), I=1,N3D)
         25 CONTINUE
      C
      C   N3D=NUMBER OF 3D-PLOTS,  H=SIZE OF TIME STEP
      C   FTIME=FINALTIME, EPSQ,EPSPTT  ARE ZERO LEVELS
      C   T3D(I)= FRACTION OF FTIME WHEN TO PLOT 3D
      C
            DO 10 I=1,N3D
            T3D(I)=FTIM*T3D(I)
         10 CONTINUE
         21 FORMAT(16I5)
         22 FORMAT(8F10.5)
            WRITE(6,28) MBP, MO, NBP, NO
         28 FORMAT(// 5X, 'M S & N S', 5X, 4I6 /)
            WRITE(6,29) AA, BB
         29 FORMAT(/ 5X, 'A & B', 9X, 2F10.3 /)
            WRITE(6,30)  H, FTIM
         30 FORMAT(/ 5X, 'INITIAL H & FINAL TIME (# LOOPS * H)', 5X, 2F10.3/)
            WRITE(6,31) N3D
         31 FORMAT(// 5X, '# OF 3-D PLOTS', 2X, I6 /)
            IF (N3D .LT. 1) GO TO 36
```

```
      WRITE(6,24) EPSQ,EPSPTT
   24 FORMAT(/ 5X,'EPSQ=',F10.5,5X,'EPSPTT=',F10.5)
      WRITE(6,32)  (T3D(I), I=1,N3D)
   32 FORMAT(/ 5X, 'AT TIME', 2X, 10F10.3 /)
   36 CONTINUE
      M01 = M0 + 1
      N01 = N0 + 1
      NS = N01 - NBP
      MS = M01 - MBP
      MSL = MS + 1
      NSL = NS  +1
      BM0 = BB * M0
      AN0 = AA * N0
      QANB = AN0 / BB
      QBMA = BM0 / AA
      8AT = -(BM0 + AN0)
      DO 47 M1 = 1,MS
      VM(M1)  = MBP + M1 - 1
   47 CONTINUE
      DO 48 N1 = 1,NS
      VN(N1)  = NBP + N1 - 1
   48 CONTINUE
      L = 0
      N3 = 1
      TPLOT = T3D(N3)
      TIME = 0.
   50 CONTINUE
      IF (TIME .GT. FTIM) GO TO 210
   52 CONTINUE
      TIME = H * L
      L = L + 1
      L8  = 0
      TIM(L)  = TIME
      IF ( L .EQ. 2) GO TO 53
      IF (TIME .LT. TPLOT) GO TO 60
C
C     SET NEXT PLOT TIME
C
      N3 = N3 + 1
      TPLOT = T3D(N3)
      IF(N3.GT.N3D) TPLOT=9999.
   53 CONTINUE
C
C     L8 = L WHEN TIME TO PLOT
C
      L8 = L
      E8T = EXP(BB * TIME) - 1.
      EAT = EXP(AA * TIME) - 1.
      AE = QANB * EBT
      BE = QBMA * EAT
```

```
      BAE = EXP(BAT * TIME)
      FJ = 1.
      JJ = 1
C
C     NO VALUE OF J = ZERO AT MO,NO
C
      FK = 1.
      KK = 1
      QFK=1.
      QFJ=1.
C
C     NO VALUE OF K = ZERO AT MO,NO
C
   60 CONTINUE
      IF (L .EQ. 1) GO TO 500
      QQ=0.
      DO 400 N1 = 1,NS
      N = NO1 - N1
      NL = NSL - N1
      NL1 = NL + 1
      ANJ = AA * N
C
C
      DO 300 M1 = 1,MS
      M = MO1 - M1
      ML = MSL - M1
      ML1 = ML + 1
      BMI = BB * M
      ABA = ANJ + BMI
      IF (M .EQ. MO) GO TO 502
      IF (M .EQ. MBP) GO TO 504
      IF (N .EQ. NO) GO TO 506
      IF (N .EQ. NBP) GO TO 507
C
C             DEFAULTS TO ALL INSIDE POINTS
C
      K1 = ANJ*P2(ML1,NL) + BMI*P2(ML,NL1) - ABA*P2(ML,NL)
      PT = ANJ*0.5* (P1(ML1,NL)+P2(ML1,NL)) + BMI*0.5 * (P1(ML,NL1) +
     1 P2(ML,NL1))
      K2 = PT - ABA * (P2(ML,NL) + H*0.5*K1)
      K3 = PT - ABA * (P2(ML,NL) + H*0.5*K2)
      K4 = ANJ*P1(ML1,NL) + BMI*P1(ML,NL1) - ABA*(P2(ML,NL) + H*K3)
      P1(ML,NL) = P2(ML,NL) + (H/6.0) * (K1+2.0*K2 + 2.0*K3 + K4)
      GO TO 200
  502 IF (N .EQ. NO) GO TO 503
      IF (N .EQ. NBP) GO TO 507
C
C             M=MO, N¬=NO,NBP
C
      K1 = BMI*P2(ML,NL1) - ABA*P2(ML,NL)
```

```
      PT = BMI*0.5 * (P1(ML,NL1) + P2(ML,NL1))
      K2 = PT - ABA * (P2(ML,NL) + H*K1*0.5)
      K3 = PT - ABA * (P2(ML,NL) + H*K2*0.5)
      K4 = BMI * P1(ML,NL1) - ABA * (P2(ML,NL) + H*K3)
      P1(ML,NL) = P2(ML,NL) + (H/6.) * (K1+2.0*K2 + 2.0*K3 + K4)
      IF (LB .EQ. L) GO TO 601
      GO TO 200
C
C             M=MO, N=NO
C
  503 CONTINUE
      P1(ML,NL) = EXP(-ABA * TIME)
      WRITE(6,102) L, N, ML,NL, P1(ML,NL)
      GO TO 200
  504 IF (N .EQ. NBP) GO TO 505
C
C
C             M=MBP, N¬= NBP
C
      K1 = ANJ * P2(ML1,NL)
      K2 = ANJ * 0.5 * (P1(ML1,NL) + P2(ML1,NL))
      K3 = K2
      K4 = ANJ * P1(ML1,NL)
      P1(ML,NL) = P2(ML,NL) + (H/6.0) * (K1 + 2.0*K2 + 2.0*K3 + K4)
      GO TO 200
C
C             M=MBP, N=NBP
C
  505 CONTINUE
      P1(ML,NL) = 0.0
      GO TO 200
C
C             N=NO, M¬=MBP, MO
C
  506 K1 = ANJ*P2(ML1,NL) - ABA*P2(ML,NL)
      PT = ANJ * 0.5 * (P1(ML1,NL) + P2(ML1,NL))
      K2 = PT - ABA * (P2(ML,NL) + H*0.5*K1)
      K3 = PT - ABA * (P2(ML,NL) + H*0.5*K2)
      K4 = ANJ * P1(ML1,NL) - ABA * (P2(ML,NL) + H*K3)
      P1(ML,NL) = P2(ML,NL) + (H/6.0) * (K1 + 2.0*K2 + 2.0*K3 + K4)
      IF (LB .EQ. L) GO TO 602
      GO TO 200
C
C             N=NBP, & N=NBP, M=MO
C
  507 CONTINUE
      K1 = BMI * P2(ML,NL1)
      K2 = BMI * 0.5 * (P1(ML,NL1) + P2(ML,NL1))
      K3 = K2
      K4 = BMI * P1(ML,NL1)
```

```
      P1(ML,NL) = P2(ML,NL) + (H/6.0) * (K1 + 2.0*K2 + 2.0*K3 + K4)
      GO TO 200
C
C              ALSO M=MO, N¬=NBP,NO
C
  601 CONTINUE
      IF((BE.LT.1.).AND.(KK.GT.9)) GO TO 255
C
C     COMPUTE PTT IF BE > 0 & KK < 10
C
      PTT=BAE
      DO 250 IND=1,KK
      XIND=IND
      PTT=PTT*BE/XIND
  250 CONTINUE
      GO TO 260
  255 PTT=0.
  260 CONTINUE
      KK = KK + 1
      IF((ABS(P1(ML,NL) -PTT)).LE.EPSPTT) GO TO 199
      GO TO 700
C
C              M=MBP, N=NO
C
  602 CONTINUE
      IF((AE.LT.1.).AND.(JJ.GT.9)) GO TO 270
C
C
C     COMPUTE PTT IF AE > 0 & JJ < 10
C
      PTT=BAE
      DO 265 IND=1,JJ
      XIND=IND
      PTT=PTT*AE/XIND
  265 CONTINUE
      GO TO 275
  270 PTT=0.
  275 CONTINUE
      JJ = JJ + 1
      IF((ABS(P1(ML,NL) -PTT)).LE.EPSPTT) GO TO 199
  700 CONTINUE
C
C     TIME REDUCED 1/2
C
      L = L - 1
      WRITE(6,701) TIME, M, N, H
  701 FORMAT(// 5X, 'H VALUE IS REDUCED BY HALF AT : TIME = ', F8.3,
     1  '  M = ', I5, '  N = ', I5, '  FROM ', F8.3 //)
      H = 0.5 * H
      WRITE(6,102) L,M,N,KK, PTT, P1(ML,NL),QFK, QFJ,QQ,QFLAG
```

```
  102 FORMAT(/ 2X, 4I6, 7F12.5)
      QFLAG=0.
      GO TO 52
  199 PTT=P1(ML,NL)
  200 CONTINUE
      QQ=QQ+P1(ML,NL)
  300 CONTINUE
  400 CONTINUE
      QQ1=ABS(QQ-1.)
      IF(QQ1.LE.EPSQ) GO TO 401
      QFLAG=1.
      GO TO 700
  401 CONTINUE
C
C     COMPUTE VARX(T), VARY(T),NBAR(T),MBAR(T),Q(T)
C
      SM8 = 0.
      SN8 = 0.
      SQ = 0.
      SM2 = 0.
      SN2 = 0.
      DO 415 M1 = 1,MS
      RM = VM(M1)
      RM2 = RM*RM
      DO 410 N1 = 1,NS
      RN = VN(N1)
      RN2 = RN*RN
      PT = P1(M1,N1)
      P2(M1,N1) = PT
      SM8 = SM8 + RM*PT
      SN8 = SN8 + RN*PT
      SQ = SQ + PT
      SM2 = SM2 + RM2 * PT
      SN2 = SN2 + RN2 * PT
  410 CONTINUE
  415 CONTINUE
      VM8(L) = SMB
      VNB(L) = SNB
      VQ(L) = SQ
      VARX(L) = SM2 - SMB*SMB
      VARY(L) = SN2 - SNB*SNB
C
C     RETURN TO MAIN LOOP (50) IF NOT TIME TO PRINT
C
      IF (N3D .LT. 1) GO TO 430
      IF(L8.NE.L) GO TO 430
      IF(L8.EQ.2) GO TO 430
C
C     ADJUST X,Y, VECTORS FOR PLOT
C
```

```
      DO 15 K=1,MS
   15 EST(K)=K-1
      DO 16 K=1,NS
      DETERM(K)=K-1
   16 CONTINUE
      DO 115 I=1,41
      CST(I)=I
      CETERM(I)=I
      DO 115 J=1,41
      PC(I,J)=0.0
  115 CONTINUE
      MP=41
       NP=41
      ALP=15.
      BETA=30.
      PMAX=P1(1,1)
      DO 116 I=1,NS
       WRITE(6,199)I,P1(1 ,I),P1(I,1 )
  199 FORMAT(' ',I5,10X,2F15.5)
  116 CONTINUE
      DO 18 I=1,MS
      DO 18 J=1,NS
      IF(P1(I,J).GT.PMAX) PMAX=P1(I,J)
   18 CONTINUE
      CONST=8.0/PMAX
      DO 19 I=1,MS
       N=I+MBP
      DO 19 J=1,NS
       M=J+NBP
      P1(I,J)=CONST*P1(I,J)
      PC(N,M)=P1(I,J)
   19 CONTINUE
      CALL PLT3D1(CST,MP,CETERM,NP,PC,ALP,BETA,F,TTL,SIZE,WK,IDN,KX,KY,N
     1KXY,LINES)
C
  430 CONTINUE
C
C
C     PLOT HERE
C     RETURN TO MAIN LOOP
C
      GO TO 50
  500 CONTINUE
      DO 501 I = 1,MS
      DO 501 J = 1,NS
      P1(I,J) = 0.0
  501 CONTINUE
      P1(MS,NS) = 1.0
      GO TO 401
C
```

```
C     FINAL TIME REACHED
C
  210 CONTINUE
      WRITE(6,211) TIME, FTIM, L
  211 FORMAT(// 5X, 'COMPUTED TIME ', F10.3, 5X, 'INPUT FINAL TIME ',
     1 F10.3, 5X, '* OF LOOPS TO REACH FINAL TIME ', I5 //)
      DO 215 I =1,L
      WRITE(6,213) VMB(I), VNB(I), VQ(I), VARX(I), VARY(I), TIM(I)
  213 FORMAT(2X, 12F10.5)
  215 CONTINUE
  650 FORMAT('1        ')
      WRITE(6,650)
      CALL PLOTP(TIM,VMB,L,0)
      WRITE(6,650)
      CALL PLOTP(TIM,VARX,L,0)
      WRITE(6,650)
      CALL PLOTP(TIM,VNB,L,0)
      WRITE(6,650)
      CALL PLOTP(TIM,VARY,L,0)
      STOP
      DEBUG SUBCHK
      END
```

APPENDIX B

```
C
C   THIS PROGRAM CALCULATES THE STATE PROBABILITIES FOR THE EQUAL
C   ATTRITION RATE COEFFICIENT SOLUTION (EARCS).
C   A  3-D PLOT IS PRODUCED USING THE VERSATEC PLOTTER.
C
      IMPLICIT REAL*8 (A-H,O-Z)
        CALL ERRSET(208,256,10,1)
      DIMENSION C(50,50),M(50),N(50),F(50),PTMN(50,50)
      REAL*4 SIZE(2),FL(2),WK(41,41,3),X(41),Y(41),P(41,41)
      DIMENSION KX(100),KY(100)
      LOGICAL*1 IDN(41,41)
      REAL*8 TTL(12)/12*'        '/
      READ(5,100)MO,MBP,NO,NBP
      READ(5,101)A
      READ(5,102)TIME
       READ(5,103) (F(I),I=1,41)
C
C   F(I)=1/I-FACTORIAL.  DONE TO SPEED UP THE PROGRAM
C
      WRITE(6,802)MO,MBP,NO,NBP,A,TIME
  802 FORMAT('  ',4I5,2F10.5)
      MD=MO-MBP
      ND=NO-NBP
      MO1=MO+1
      NO1=NO+1
      MBP1=MBP+1
      NBP1=NBP+1
      MBP2=MBP+2
      NBP2=NBP+2
      RMO=MO
      RNO=NO
      DO 10 I=1,41
      DO 10 J=1,41
      C(I,J)=0.0
      PTMN(I,J)=0.0
       P(I,J)=0.0
   10  CONTINUE
      C(MO1,NO1)=1.0
      DO 20 I=2,MD
      J=MBP+I
      C(J,NO1)= RNO**(MD-I+1)
   20  CONTINUE
      DO 21 I=2,ND
      J=NBP+I
      C(MO1,J)= RMO**(ND-I+1)
   21  CONTINUE
      DO 22 I=2,MD
      MC=MO1-I
      MM=MO1-I+1
      MPLUS=MM+1
```

```
      DO 22 J=2,ND
      NC=N01-J
      NN=N01-J+1
      NPLUS=NN+1
      C(MM,NN)=NC*  C(MPLUS,NN)+MC*  C(MM,NPLUS)
   22  CONTINUE
C
C END OF COEFFICIENT CALCULATION
C
       F1=-A*TIME
      DO 25 I=MBP2,M01
      DO 25 J=NBP2,N01
      IJ=I+J
      K=M0+N0-IJ +2
      F2=(1.0-DEXP(F1))**K
      KK=K+1
      IJ=IJ-2
      F3=DEXP(F1*IJ)
      PTMN(I,J)=F2*F3*C(I,J)*F(KK)
   25  CONTINUE
  100  FORMAT(4I5)
  101  FORMAT(F10.5)
  102  FORMAT(F10.5)
  103  FORMAT(D17.11)
C
C FOR NBP<N<N0
C
      DO 30 II=1,ND
      NFORCE=NBP+II
      J=M0+N0-MBP-1-NFORCE
      FACT=MBP+1.0+NFORCE
      SUM1=(1.0-DEXP(F1*FACT))/( A*FACT)
      SUMINT=SUM1
      DO 301 K=1,J
      FACT=FACT+1.0
      ADDFAC=(-1.0)**K
      FRACTN=(1.0-DEXP(F1*FACT))/( A*FACT)
      COMBT=1.0
      DO 302 KJ=1,K
      RKJ=KJ
      COMBI=(J-RKJ+1.0)/RKJ
      COMBT=COMBT*COMBI
  302  CONTINUE
      SUMINT=SUMINT+ADDFAC*FRACTN*COMBT
  301  CONTINUE
      NMBP=J+1
      NBOUND=NFORCE+1
      PFAC=A*NFORCE*F(NMBP)*C(MBP2,NBOUND)
      PTMN(MBP1,NBOUND)=PFAC*SUMINT
   30  CONTINUE
```

```
C
C FOR MBP<M<MO
C
      DO 40 II=1,MO
      MFORCE=MBP+II
      J=MO+NO-NBP-1-MFORCE
      FACT=NBP+1.0+MFORCE
      SUM1=(1.0-DEXP(F1*FACT))/( A*FACT)
      SUMINT=SUM1
      DO 401 K=1,J
      FACT=FACT+1.0
      ADDFAC=(-1.0)**K
      FRACTN=(1.0-DEXP(F1*FACT))/( A*FACT)
      COMBT=1.0
      DO 402 KJ=1,K
      RKJ=KJ
      COMBI=(J-RKJ+1.0)/RKJ
      COMBT=COMBT*COMBI
  402  CONTINUE
      SUMINT=SUMINT+ADDFAC*FRACTN*COMBT
  401  CONTINUE
      MNBP=J+1
      MBOUND=MFORCE+1
      PFAC=A*MFORCE*F(MNBP)*C(MBOUND,NBP2)
      PTMN(MBOUND,NBP1)=PFAC*SUMINT
   40  CONTINUE
      DO 45 I=MBP1,MO1
      DO 45 J=NBP1,NO1
      K=I-1
      L=J-1
      WRITE(6,801)K,L,PTMN(I,J)
  801 FORMAT(' P(T,', I5,',',I5,') = ',D17.11)
   45 CONTINUE
C
C DATA ADJUSTMENT FOR PLOT
C
      NROW=41
      NCOL=41
      NKXY=100
      LINES=0
      ALPHA=15.
      BETA=30.
      FL(1)=0.0
      FL(2)=0.0
      SIZE(1)=6.0
      SIZE(2)=8.0
C
C SCALING
C
      PMAX=PTMN(1,1)
```

```
      DO 50 I=1,41
      X(I)=I
      DO 50 J=1,41
      Y(J)=J
      IF(PTMN(I,J).GT.PMAX)PMAX=PTMN(I,J)
   50  CONTINUE
      CONST=8.0/PMAX
      DO 51 I=1,41
      DO 51 J=1,41
      P(I,J)  = SNGL(CONST*PTMN(I,J))
   51  CONTINUE
C
C
C IF  OTHER FORCE LEVELS CHANGE PLOT ARGUMENTS
C
      CALL PLT3D1(X,NROW,Y,NCOL,P,ALPHA,BETA,FL,TTL,SIZE,WK,
     *            ION,KX,KY,NKXY,LINES)
C
C  EXPECTED VALUES AND  VARIANCES
C
      EM=0.0
      EMM=0.0
      VARM=0.0
       SPROB=0.0
      DO 60 II=MBP1,M01
      I=II-1
      SPT=0.0
      DO 601 JJ=NBP1,N01
      SPT=SPT+  PTMN(II,JJ)
  601  CONTINUE
      EM=EM+I*SPT
      EMM=EMM+ I*I*SPT
      SPROB=SPROB+SPT
   60 CONTINUE
      VARM=EMM-EM*EM
      EN=0.0
      ENN=0.0
      VARN=0.0
      DO 70 II=NBP1,N01
      I=II-1
      SPT=0.0
      DO 701 JJ=MBP1,M01
      SPT=SPT+  PTMN(JJ,II)
  701  CONTINUE
      EN=EN+I*SPT
      ENN=ENN+ I*I*SPT
   70 CONTINUE
      VARN=ENN-EN*EN
      WRITE(6,805) EM,EN,VARM,VARN
  805 FORMAT(' ','EXPECTED VALUES M , N',  50X,' VARIANCE M,N ',/
```

```
     *' ',D17.11,10X, D17.11,T51,D17.11, 10X, D17.11)
      WRITE(6,1100) SPROB
 1100 FORMAT(' SUM OF PROBABILITIES ',D17.11)
      STOP
      END
```

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